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Nevanlinna's main theorems on Riemann surfaces

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Introduction

Our interest lies in Nevanlinna's main theorems for an arbitrary analytic mapping of an open Riemann surface S of parabolic type into a closed Riemann surface R. We may regard S as a covering surface of R.

The present paper is originated from the lectures given at Hiroshima University in 1974-75. The author tried there to understand the fundamental paper [1] of Ahlfors. As shown in our Appendix 1 one needs some modification of Ahlfors' discussions. Aside from this point we follow fairly faithfully his paper. We start with

$$h(P'; P, \mu) = \int h(P'; P, Q)d\mu(Q)$$

instead, as Chern [4] did (see our Appendix 2), of a solution s of the equation $\Delta s = 2\pi\rho^2$ (ρ: density), where $h(P'; P, Q)$ is a harmonic function of P' on R with positive (negative resp.) logarithmic singularity at P (Q resp.).

In §2 we prove the first main theorem for a general non-negative measure μ, which may not have a density. This is the only essentially new result in our paper. As shown in Theorem 3 the difference of two characteristics $T(r)$ with respect to different measures is bounded so that it is our disposal which measure we choose. Evidently there exists a measure with positive density ρ on any closed Riemann surface. The choice of such a ρ simplifies the matters, although we prove an important identity also for ρ with zero points in Appendix 5. There are two ways to obtain such an identity. In the text we use, as Sario did, a classical relation concerning the characteristic of a domain, and in Appendix 4, as Ahlfors did, we apply Gauss-Bonnet's formula.

As stated in [1; p.10] there are two ways to derive the second main theorem. In the text we follow like Ahlfors the way which is not usually chosen. The usual way is presented in Appendix 6. After discussions on defect relation in §5, a detailed proof of Ahlfors' disk theorem in [1; §4] is proved in the last section.

In addition to the appendices mentioned above, we are concerned with double integrals $\int_o^r \int_s^r T(t)dtds$, etc. instead of $T(r)$, etc. in Appendix 7, and give a proof of coarea formula in Appendix 8 to make our paper self-contained.

Before closing our introduction we indicate some problems. In our paper it is investigated how isolated points or disks are covered by the covering surface S. There remains the problem to see how an arbitrary set in R is covered by S.

Riemann surfaces are naturally two-dimensional. It might be possible to generalize the first main theorem to mappings of spaces of higher dimensions which preserve harmonicity like Fuglede's harmonic morphisms in [5]. It would be of some interest to find, not only from value distribution theoretic point of view but also from purely potential theoretic point of view, properties of pull backs of harmonic and superharmonic functions.

Because of the limited time for preparation of the manuscript there may be incompleteness in presentation of the paper and in proofs of theorems. The author hopes nevertheless that this informal paper serves as a base for further progress of the theory.

§1. Function h

Let R be a Riemann surface. It is called hyperbolic if a Green's function exists on it. Otherwise it is called parabolic. First we prove

Lemma 1. Let R be a hyperbolic Riemann surface. If P_1, P_2, ... $\to P_0$, then $g(P, P_n) \to g(P, P_0)$ uniformly outside any open neighborhood V of P_0.

Proof. Let Δ: $|z| \leqq 1$ correspond to a closed disk $V_0 \subset V$ on R with center at P_0. We may assume that V_0 contains all P_n in its interior. Let z_n be the image of P_n. We write

$$h_n(z) = \log \left|\frac{1 - \bar{z}_n z}{z - z_n}\right| \quad \text{and} \quad h(z) = \log \frac{1}{|z|}.$$

Denote the harmonic measure of ∂V_0 with respect to $R - V_0$ by ω. Let us see that $0 < \omega < 1$. Set $m = \min g(P, P_0)$ on ∂V_0. Then $g(P, P_0)/m \geqq \omega$ on $R - V_0$. Since $\inf g = 0$, $\inf \omega = 0$, and since $\omega = 1$ on ∂V_0, $0 < \omega < 1$ on $R - V_0$.

For a small $\varepsilon > 0$ set $M = \max_{|z|=1+\varepsilon} \omega(P(z))$ and

$$a_n = \frac{\max\limits_{|z|=1+\varepsilon} (h(z) - h_n(z))}{1 - M}.$$

Then

$$a_n - a_n\omega \geqq h - h_n \qquad \text{on } 1 \leqq |z| \leqq 1 + \varepsilon$$

by the maximum principle. Denote by $\mathcal{V}(P)$ the family of positive continuous superharmonic functions v on $R - \{P\}$ such that $v + \log |z|$ is superharmonic on an open disk corresponding to $|z| < 1$, where P corresponds to $z = 0$. For any $v \in \mathcal{V}(P_0)$ the function equal to

$v + h_n - h + a_n$ on V_0 and to $v + a_n\omega$ on $R - V_0$ belongs to $V(P_n)$ so that

$$v + a_n\omega \geqq g(\cdot, P_n) \qquad \text{on } R - V_0.$$

The arbitrariness of v yields

$$g(\cdot, P_n) - g(\cdot, P_0) \leqq a_n\omega \qquad \text{on } R - V_0.$$

Similarly

$$g(\cdot, P_0) - g(\cdot, P_n) \leqq b_n\omega \qquad \text{on } R - V_0$$

with

$$b_n = \frac{\max\limits_{|z|=1+\varepsilon} (h_n(z) - h(z))}{1 - M}.$$

Since $a_n, b_n \to 0$ as $n \to \infty$, $g(P, P_n) \to g(P, P_0)$ uniformly on $R - V_0$. This proves our lemma.

Let R be parabolic. For $P_1, P_2 \in R$ let $h = h(P; P_1, P_2)$ be a function harmonic on $R - \{P_1, P_2\}$, bounded outside any neighborhood of P_1 and P_2 and having singularities of the form $-\log |z|$ and $\log |z|$ at P_1 and P_2 respectively. It is determined up to an additive constant. As to the existence see, for instance, [9; Theorem 2.2 and Chap. II, 4].

Lemma 2. Let R be a parabolic Riemann surface and fix P_0 on R. Let z be a local parameter on a disk with center at P_0. If $P_1, P_2, \ldots \to P_0$ and $Q_1, Q_2, \ldots \to Q_0 \neq P_0$, then $h(P; P_k, Q_k) \to h(P; P_0, Q_0)$ locally uniformly on $R - \{P_0, Q_0\}$, where $h(P; P_0, Q_0)$ is normalized in such a way that

$$h(P(z); P_0, Q_0) + \log |z| \to 0 \qquad \text{as } z \to 0$$

and every $h(P; P_k, Q_k)$ is normalized in such a way that

$$h(P(z); P_k, Q_k) + \log |z - z(P_k)| \to 0 \qquad \text{as } z \to z(P_k). \tag{1}$$

Proof. Choose two arbitrarily small open disks U_1 and U_2 with centers at P_0 and Q_0 respectively, and denote their union by U. We assume that all P_k and Q_k are contained in U_1 and U_2 respectively. Let V be an arbitrary closed disk lying in the exterior of U. Set

$$u_k(P) = g_{R-V}(P, P_k) - g_{R-V}(P, Q_k).$$

It converges to $g_{R-V}(P, P_0) - g_{R-V}(P, Q_0)$ uniformly on ∂U by Lemma 1. Hence $|u^k| < \alpha < \infty$ on ∂U.

We consider

$$v_k(P) = h(P; P_k, Q_k) - \max_{\partial U} h(\cdot; P_k, Q_k).$$

Then max $v_k = 0$ on ∂U. If there were $P^* \in R - U$ with $v_k(P^*) > 0$, the function equal to max $(v_k, v_k(P^*)/2)$ on $R - U$ and to $v_k(P^*)/2$ on U would be non-constant subharmonic and bounded above on R. This contradicts the parabolicity of R. Thus $v_k \leqq 0$ on $R - U$. Since $v_k - u_k$ is bounded harmonic on $R - V$,

$$\begin{aligned} 0 = \max_{\partial U} v_k &\leqq \max_{\partial U} (v_k - u_k) + \max_{\partial U} u_k \\ &\leqq \max_{\partial V} (v_k - u_k) + \max_{\partial U} u_k = \max_{\partial V} v_k + \max_{\partial U} u_k. \end{aligned}$$

Thus $\max_{\partial V} v_k \geqq -\max_{\partial U} u_k > -\alpha$. Since $v_k \leqq 0$ on $R - U$, there is a subsequence $\{v_{k_j}\}$ converging to a harmonic function v or to $-\infty$ locally uniformly on $R - U \cup \partial U$. The latter case does not happen because $\max_{\partial V} v_k > -\alpha$. If $|z| < 1$ corresponds to U_1 and z_k to P_k, then $v_{k_j} + \log |z - z_{k_j}|$ is harmonic on $|z| \leqq 1 + \varepsilon$ for small $\varepsilon > 0$ and converges uniformly on $|z| = 1 + \varepsilon$ and hence on $U_1 \cup \partial U_1$. Thus v_{k_j} converges to a harmonic function also on $U_1 \cup \partial U_1 - \{P_0\}$.

Similarly it converges on $U_2 \cup \partial U_2 - \{Q_0\}$. We denote the limit by v again. It is equal to $h(\cdot;\ P_0,\ Q_0) + \text{const}$. We note that the value of $v_{k_j} + \log|z - z_{k_j}|$ at $z = z_{k_j}$ is equal to $-\max_{\partial U} h(\cdot;\ P_{k_j},\ Q_{k_j})$ and tends to the value of $v + \log|z|$ at $z = 0$. Hence

$$h(\cdot;\ P_0,\ Q_0) = v + \lim_{j\to\infty} \max_{\partial U} h(\cdot;\ P_{k_j},\ Q_{k_j})$$

$$= \lim_{j\to\infty} \{v_{k_j} + \max_{\partial U} h(\cdot;\ P_{k_j},\ Q_{k_j})\}$$

$$= \lim_{j\to\infty} h(\cdot;\ P_{k_j},\ Q_{k_j})$$

locally uniformly on $R - \{P_0,\ Q_0\}$. Since from any subsequence of $\{h(P;\ P_k,\ Q_k)\}$ we can extract a subsequence converging to $h(P;\ P_0,\ Q_0)$ locally uniformly on $R - \{P_0,\ Q_0\}$, $h(P;\ P_k,\ Q_k) \to h(P;\ P_0,\ Q_0)$ locally uniformly on $R - \{P_0,\ Q_0\}$. Our lemma is now proved.

Lemma 3. Let R be a closed Riemann surface, and $P_0',\ P_0,\ Q_0$ be mutually different. If $P_1',\ P_2',\ \ldots \to P_0',\ P_1,\ P_2,\ \ldots \to P_0$ and $Q_1,\ Q_2,\ \ldots \to Q_0$, then $h(P_k';\ P_k,\ Q_k) \to h(P_0';\ P_0,\ Q_0)$, where every $h(P;\ P_k,\ Q_k)$ is normalized at P_k as (1).

Proof. Take a compact set K which contains $P_1',\ P_2',\ \ldots$ but not $\{P_0,\ Q_0\}$. By Lemma 2 $h(P';\ P_k,\ Q_k) \to h(P';\ P_0,\ Q_0)$ uniformly on K. Hence, given $\varepsilon > 0$, there exists k_0 such that

$$|h(P';\ P_k,\ Q_k) - h(P';\ P_0,\ Q_0)| < \frac{\varepsilon}{2} \quad \text{on } K$$

if $k \geqq k_0$. In particular,

$$|h(P_k';\ P_k,\ Q_k) - h(P_k';\ P_0,\ Q_0)| < \frac{\varepsilon}{2}$$

if $k \geqq k_0$. Since $h(P; P_0, Q_0)$ is continuous outside $\{P_0, Q_0\}$, there exists $k_1 > k_0$ such that

$$|h(P_k'; P_0, Q_0) - h(P_0'; P_0, Q_0)| < \frac{\varepsilon}{2}$$

if $k \geqq k_1$. There follows

$$|h(P_k'; P_k, Q_k) - h(P_0'; P_0, Q_0)| < \varepsilon$$

for $k \geqq k_1$. Thus $h(P_k'; P_k, Q_k) \to h(P_0'; P_0, Q_0)$ as $k \to \infty$.

§2. First main theorem

Let R be a closed Riemann surface, S be an arbitrary Riemann surface and $f = f(\tilde{P})$ be a non-constant analytic mapping of S into R. Then S may be regarded as a covering surface of R. Let S_0 be a closed disk in S such that the boundary ∂S_0 does not contain any branch point of S, and G be a relatively compact subdomain of S which includes S_0 and whose boundary consists of finitely many analytic closed curves. Let u_G be the harmonic function in $G - S_0$ which is equal to 0 on ∂S_0 and to a constant c_G on ∂G and for which $\int_{\partial S_0} \partial u_G/\partial n ds = 1$. For any $t \in [0, c_G)$ denote by γ_t the level curve $u_G = t$, and by G_t the domain $\{0 < u_G < t\} \cup S_0$.

Let P be a point of R, and $\tilde{P}_1, \tilde{P}_2, \ldots$ be the inverse images of P in G_t. Denote their numbers, counted with multiplicity, by $n(t, P)$, and set $N(r, P) = \int_0^r n(t, P)dt$.

Normalize $h(P'; P, Q)$ as (1). By Green's formula we see easily that $h(P'; P, Q) = h(Q; P, P')$. Hence it is natural to define $h(P'; P, P) = \infty$. We set

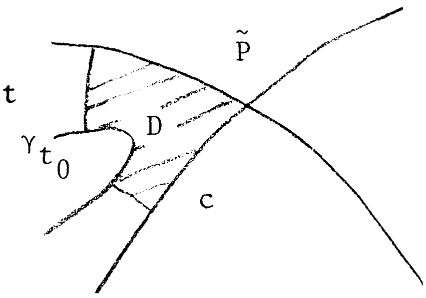
$$h(P'; P, \mu) = \int h(P'; P, Q) d\mu(Q)$$

for a non-negative measure μ on R. We assume $h(P'; P, \mu) \not\equiv \infty$. Let B be a Borel set on S. If B is contained in a disk on S which is homeomorphic to its projection, then set $\tilde{\mu}(B) = \mu(f(B))$. If B consists of a branch point $\tilde{P}$ with multiplicity n, then set $\tilde{\mu}(\{P\}) = n\mu(\{f(\tilde{P})\})$. In this way we obtain the pull back $\tilde{\mu}$ of μ on S.

We prove first

Lemma 4. Fix $\tilde{P} \in G$, and let $g(\cdot, \tilde{P})$ be the Green function with pole $\tilde{P}$ on a domain containing G. Then $\int_{\gamma_t} g(\cdot, \tilde{P}) \partial u_G/\partial n ds \to \int_{\gamma_r} g(\cdot, \tilde{P}) \partial u_G/\partial n ds$ as $t \uparrow r$, where $0 < r < c_G$.

Proof. It is sufficient to consider the case $\tilde{P} \in \gamma_r$. Define a conjugate harmonic function u_G^* in a neighborhood of $\tilde{P}$ so that $u_G^*(\tilde{P}) = 0$. Suppose there are 2p ($p \geqq 1$) branches of γ_r issuing from $\tilde{P}$. We find p arcs such that each arc consists of two branches and G_r lies on one side of each arc.



Let c be such an arc on which $-\delta < u_G^* < \delta$. The shaded part D in the figure is a domain bounded by c, two arcs on each of which u_G^* is constant and a part of γ_{t_0} for $t_0 < r$. Denote by $c_{t,\delta}$ ($t_0 < t < r$) the subarc of γ_t lying in D. Let $\varepsilon > 0$ be given. It suffices to show that

$$\left| \int_{c_{t,\delta}} g \, du_G^* \right| < \varepsilon$$

for every t, $t_0 < t < r$ if δ and $r - t_0$ are small. Set

$$F = u_G + iu_G^* - u_G(\tilde{P}).$$

We may take

$$w = F^{1/p}$$

as a local parameter around $\tilde{P}$. Then

$$g(\tilde{Q}, \tilde{P}) = \log \frac{1}{|w|} + \text{a continuous function } G(w)$$

$$= -\frac{1}{p} \log |F| + G(w).$$

Let $|G| < M < \infty$ on $\cup_{t_0<t<r} c_{t,\delta}$ for some $t_0 < r$. Since

$$\left|\int_{-\delta}^{\delta} \log |u_G + iu_G^* - u_G(\tilde{P})| du_G^*\right| \leqq \int_{-\delta}^{\delta} \log \frac{1}{|\xi|} d\xi,$$

$$\left|\int_{c_{t,\delta}} g du_G^*\right| \leqq \frac{1}{p}\int_{-\delta}^{\delta} \log \frac{1}{|\xi|} d\xi + O(\delta) < \varepsilon$$

if δ is small. This proves our lemma.

Lemma 5. Let ϕ be a function of class C^∞ with compact support in a plane. Let U^μ be a logarithmic potential. Then

$$\frac{1}{2\pi}\int_{\partial D} \frac{\partial(U^\mu * \phi)}{\partial n} ds = -(\phi * \mu)(D)$$

for any domain D with smooth boundary.

Proof. We have

$$(U^\mu*\phi)(z) = \iint U^\mu(z-\zeta)\phi(\zeta)d\xi d\eta = \iint \phi(\zeta)d\xi d\eta\int \log \frac{1}{|z-\zeta-\omega|} d\mu(\omega)$$

$$= \iint \log \frac{1}{|w|} dudv\int \phi(z-\omega-w)d\mu(\omega)$$

$$= \iint (\phi*\mu)(z-w)\log \frac{1}{|w|} dudv = U^{\phi*\mu}(z)$$

and

$$\frac{1}{2\pi}\int_{\partial D} \frac{\partial(U^{\mu}*\phi)}{\partial n}\, ds = -\ (\phi*\mu)(D).$$

This proves our lemma.

In general, let v be a superharmonic function in a subdomain S' of S. Locally v is expressed as the sum of a logarithmic potential and a harmonic function. We call the non-negative measure which gives the logarithmic potential the measure locally associated with v. We obtain the global measure on S' by means of the measure locally associated with v, and call it the measure associated with v.

Lemma 6. Let $G \subset S$ be as above and $\tilde{P}_1, \ldots, \tilde{P}_q$ be points of G. Let v be a subharmonic function on $G - \{\tilde{P}_1, \ldots, \tilde{P}_q\}$ which is harmonic in a punctured disk around each $\tilde{P}_i$ and which has a logarithmic singularity of the form $a_i \log |z|$ at $\tilde{P}_i$. Let μ_v be the measure associated with -v in $G - \{\tilde{P}_1, \ldots, \tilde{P}_q\}$. Set $a(t) = \sum_i' a_i$ where the summation extends over $\tilde{P}_i$ which are contained in G_t. Then, writing $\gamma_r - \gamma_0$ for $\gamma_r \cup \gamma_0^{-1}$,

$$\frac{1}{2\pi}\int_{\gamma_r-\gamma_0} v du_G^* = \int_0^r \mu_v(G_t)dt + \int_0^r a(t)dt.$$

Proof. Fix any domain $G_0 \supset G \cup \partial G$ relatively compact in S. Consider the Green function $g(\tilde{P}, \tilde{Q})$ on G_0, and set

$$U(\tilde{P}) = \int g(\tilde{P}, \tilde{Q})d\mu_v(\tilde{Q}).$$

Then $\mu_v = \mu_{-U}$. The function

$$h_0(\tilde{P}) = v(\tilde{P}) + U(\tilde{P}) + \sum_{i=1}^{q} a_i g(\tilde{P}, \tilde{P}_i)$$

is harmonic on G. We have

$$\int_{\gamma_t} \frac{\partial h_0}{\partial t}\, du_G^* = 0 \qquad \text{for any } t,\ 0 \leqq t \leqq c_G,$$

and hence $\int_{\gamma_r - \gamma_0} h_0 du_G^* = 0$ for any r, $0 < r < c_G$. We have also

$$\frac{1}{2\pi}\int_{\gamma_t} \frac{\partial g(\cdot,\ \tilde{P}_i)}{\partial t}\, du_G^* = \begin{cases} -1 & \text{if } \tilde{P}_i \in G_t, \\ 0 & \text{if } \tilde{P}_i \notin G_t \cup \gamma_t, \end{cases}$$

and hence

$$\frac{1}{2\pi}\int_{\gamma_r - \gamma_0} \sum_{i=1}^{q} a_i g(\cdot,\ \tilde{P}_i) du_G^* = -\int_0^r a(t)dt$$

if no $\tilde{P}_i$ lies on $\gamma_r \cup \gamma_0$. By Lemma 4 one sees that the relation holds in general.

To completes the proof it is sufficient to establish

$$(2) \qquad \frac{1}{2\pi}\int_{\gamma_r - \gamma_0} U du_G^* = -\int_0^r \mu_{-U}(G_t)dt.$$

First we assume that grad u_G does not vanish on the support of μ_v. Take $r_n \downarrow r$, and consider the Green potential U_n of the restriction of μ_v to $\{r_n \leqq u_G < r_{n-1}\}$; the potential of the restriction of μ_v to $\{r_1 \leqq u_G < c_G\}$ is denoted by U_1. We have $\int_{\gamma_t} \partial U_n/\partial t du_G^* = 0$ for any $t \in (0,\ r)$ and hence $\int_{\gamma_r - \gamma_0} U_n du_G^* = 0$. Hence we may assume from the beginning that the support of μ_v is contained in $G_r \cup \partial G_r$. By using a partition of unity we may assume that the support of μ_v is contained in a domain D of the form $\{t_1 < u_G < t_2,\ s_1 < u_G^* < s_2\}$. We may assume also that grad $u_G \neq 0$ on its closure. Fix $\tilde{P} \in D$ and take $z = u_G + iu_G^* - u_G(\tilde{P})$ with $u_G^*(\tilde{P}) = 0$ as a local parameter on D. Let $\psi_n(\tau) \geqq 0$ be a non-negative function on $0 \leqq$

$\tau \leqq \infty$ such that $\psi_n = 0$ on $1/n \leqq \tau < \infty$, $\phi_n(z) = \phi_n(x, y) = \psi_n(\sqrt{x^2 + y^2}) \in C^\infty$ and $\iint \phi_n dxdy = 1$. By Lemma 5

$$\frac{1}{2\pi}\int_{\partial(G_t \cap D)} \frac{\partial(U*\phi_n)}{\partial n} ds = -(\phi_n * \mu_V)(G_t \cap D) = -(\phi_n * \mu_V)(G_t).$$

Since U is harmonic on ∂D, $U*\phi_n = U$ on ∂D if n is large. Therefore

$$\int_{\partial(G_t \cap D)} \frac{\partial(U*\phi_n)}{\partial n} ds = \int_{\gamma_t \cap D} \frac{\partial(U*\phi_n)}{\partial t} du_G^* + \int_{\gamma_t - D} \frac{\partial U}{\partial t} du_G^*.$$

By integration

$$\frac{1}{2\pi}\int_{(\gamma_{t_2} - \gamma_{t_1}) \cap \partial D} U*\phi_n du_G^* + \frac{1}{2\pi}\int_{(\gamma_{t_2} - \gamma_{t_1}) - D} U du_G^* = -\int_{t_1}^{t_2} (\phi_n * \mu_V)(G_t) dt.$$

Letting $n \to \infty$ we derive

$$\frac{1}{2\pi}\int_{\gamma_{t_2} - \gamma_{t_1}} U du_G^* = -\int_{t_1}^{t_2} \mu_V(G_t) dt.$$

Since U is harmonic on $\{0 \leqq u_G \leqq t_1\}$ and $\{t_2 \leqq u_G \leqq r\}$, it is easy to see that

$$\frac{1}{2\pi}\int_{\gamma_r - \gamma_{t_2}} U du_G^* = -\int_{t_2}^{r} \mu_V(G_t) dt \ (= -\mu_V(G_{t_2})(r - t_2))$$

and

$$\frac{1}{2\pi}\int_{\gamma_{t_1} - \gamma_0} U du_G^* = -\int_0^{t_1} \mu_V(G_t) dt \quad (= 0).$$

Accordingly (2) is derived.

Lastly we consider the case when grad u_G vanishes at some points of the support of μ_V. Let $\tilde{Q}_1, \ldots, \tilde{Q}_n$ be the zero points of grad u_G on $G_r \cup \gamma_r$ and assume $\mu_V(\{\tilde{Q}_1, \ldots, \tilde{Q}_n\}) = 0$. Suppose

$|z_i| < 1$ corresponds to a disk on S around $\tilde{Q}_i$ for each i, $1 \leqq i \leqq n$, and denote by $D_i^{(m)}$ the image of $|z_i| < 1/m$ on S. Denote by μ_m the restriction of μ_v to $S - \cup_i D_i^{(m)}$, and by $U^{(m)}$ the Green potential of μ_m. We have (2) for $U^{(m)}$. By letting $m \to \infty$ we obtain (2). Suppose $\mu_v(\{\tilde{Q}_1, \ldots, \tilde{Q}_n\}) > 0$. Set $b_i = \mu_v(\{\tilde{Q}_i\})$ and $U_0 = U - \sum_i b_i g(\cdot, \tilde{Q}_i)$. Evidently $\mu_{-U_0}(\{\tilde{Q}_1, \ldots, \tilde{Q}_n\}) = 0$. Hence (2) is true for U_0. We have already seen that (2) is true for $g(\cdot, \tilde{Q}_i)$. Thus (2) is true for general U. Our lemma is now proved.

We shall establish the following first main theorem.

Theorem 1. We have

$$\mu(R)N(r, P) + \frac{1}{2\pi}\int_{\gamma_r - \gamma_0} h(f(\tilde{P}); P, \mu)du_G^*(\tilde{P}) = \int_0^r \tilde{\mu}(G_t)dt. \tag{3}$$

Proof. From our assumption $h(P'; P, \mu) \not\equiv \infty$ it follows that $\mu(\{P\}) = 0$. First we assume $P \notin S_\mu$ (= the support of μ). Regard $h(f(\tilde{P}); P, \mu)$ as a function on G and denote it by v. Then the measure associated with v is equal to the pull back $\tilde{\mu}$. The condition in Lemma 6 is satisfied with $\{\tilde{P}_1, \ldots, \tilde{P}_q\} = \{\tilde{P} \in G; f(\tilde{P}) = P\}$. The singularity of v at $\tilde{P}_i$ has the form $-n_i\mu(R)\log |z|$, where n_i is the multiplicity of f at $\tilde{P}_i$. It follows that $\sum n_i = n(t, P)$. By Lemma 6 we have

$$\frac{1}{2\pi}\int_{\gamma_r - \gamma_0} h(f(\tilde{P}); P, \mu)du_G^*(\tilde{P}) = \int_0^r \tilde{\mu}(G_t)dt - \mu(R)\int_0^r n(t, P)dt.$$

Next consider the case $P \in S_\mu$. Suppose $|w| < 1$ corresponds to a disk on R with center at P and denote by D_m the image of

$|w| < 1/m$. Denote by μ_m the restriction of μ to $R - D_m$. We have

$$\frac{1}{2\pi}\int_{\gamma_r-\gamma_0} h(f(\tilde{P});\ P,\ \mu_m)du_G^*(\tilde{P}) = \int_0^r \tilde{\mu}_m(G_t)dt - \mu_m(R)\int_0^r n(t,\ P)dt.$$

By letting $m \to \infty$ we derive the required relation.

Remark. The left hand side of the identity in Theorem 1 does not depend on the choice of P, while the right hand side of

$$\mu(R)N(r,\ P) = \int_0^r \tilde{\mu}(G_t)dt - \frac{1}{2\pi}\int_{\gamma_r-\gamma_0} h(f(\tilde{P});\ P,\ \mu)du_G^*(P)$$

does not depend on the choice of μ.

Next we establish Shimizu-Ahlfors relation.

Theorem 2. $\displaystyle\int_R N(r,\ P)d\mu(P) = \int_0^r \tilde{\mu}(G_t)dt.$

Proof. First we show that $\tilde{\mu}(G_t) = \int_R n(t,\ P)d\mu(P)$. We decompose G_t into mutually disjoint Borel sets $\{B_j\}$ and branch points $\{\tilde{P}_k\}$ with multiplicities $\{n_k\}$ such that f is one-to-one on each B_j. Then

$$\tilde{\mu}(G_t) = \sum_j \tilde{\mu}(B_j) + \sum_k n_k\mu(\{f(\tilde{P}_k)\}) = \sum_j \int \chi_{f(B_j)}d\mu + \sum_j \int n_j\chi_{\{f(\tilde{P}_j)\}}d\mu$$

$$= \int_R n(t,\ P)d\mu(P),$$

where χ indicates the characteristic function. Next, we observe that $n(t,\ P)$ is lower semicontinuous on $(0,\ c_G) \times R$. Therefore one can apply Fubini's theorem and has

$$\int_R N(r,\ P)d\mu(P) = \int_R\int_0^r n(t,\ P)dtd\mu(P)$$

$$= \int_0^r dt \int_R n(t, P)d\mu(P) = \int_0^r \tilde{\mu}(G_t)dt.$$

By integrating (3) with respect to μ we obtain

Corollary. $\int_R \int_{\gamma_r - \gamma_0} h(f(\tilde{P}); P, \mu) du_G^*(\tilde{P}) d\mu(P) = 0.$

Let μ be a non-negative measure on R. We shall say that locally the logarithmic potential of μ is bounded if, on every closed parametric disk $|z| \leqq r_0$,

$$\int_{|\zeta| \leqq r_0} \log \frac{1}{|z-\zeta|} d\mu(P(\zeta))$$

is bounded as a function of z.

We shall prove

Lemma 7. Let μ be a non-negative measure on R such that locally the logarithmic potential of μ is bounded. Let Δ be a disk corresponding to $|z| \leqq r_0$ and Δ' correspond to $|z| \leqq r_0/2$. Then $h(P'; P, \mu)$ is bounded with respect to $(P', P) \in \Delta' \times (R - \Delta)$, and

$$h(P'; P, \mu) - \mu(R) \log \frac{1}{|z(P') - z(P)|}$$

is bounded with respect to $(P', P) \in \Delta \times \Delta$.

Proof. Assume that $|h(P_n'; P_n, \mu)| \to \infty$ as $n \to \infty$ for P_1', P_2', $\ldots \in \Delta'$ and $P_1, P_2, \ldots \in R - \Delta$. We assume moreover that P_n converges to P_0; this belongs to $R - (\Delta - \partial\Delta)$. Take a closed disk V in $R - \Delta'$ with center at P_0. For $P \in V$ we divide the integral $\int_R h(P'; P, Q)d\mu(Q)$ into those on Δ', V, $R - \Delta' - V$ and denote them by $I_i(P', P)$, $i = 1, 2, 3$, respectively. We have

$$I_1(P', P) = \int_{\Delta'} \{h(P'; P, Q) - \log |z(P') - z(Q)|\} d\mu(Q)$$

$$+ \int_{\Delta'} \log |z(P') - z(Q)| d\mu(Q).$$

By our assumption the last integral is bounded. Denote the integrand of the first integral by $k(P', P, Q)$. Choose ε, $0 < \varepsilon < r_0/2$, so that the image Δ'_ε of $|z| \leqq r_0/2 + \varepsilon$ is disjoint from V. We see that k is continuous with respect to (P', P, Q) on $\partial\Delta' \times V \times \Delta'$ on account of Lemma 3. Let $|k| < M < \infty$ there. Since k is a harmonic function of P' on Δ' for every fixed (P, Q) on $V \times \Delta'$, $|k(P', P, Q)| < M$ on $\Delta' \times V \times \Delta'$. Thus I_1 is bounded on $\Delta' \times V$.

Secondly, we write

$$I_2(P', P) = \int_V \{h(P'; P, Q) + \log |\zeta(P) - \zeta(Q)|\} d\mu(Q)$$

$$- \int_V \log |\zeta(P) - \zeta(Q)| d\mu(Q),$$

where $|\zeta| \leqq 1$ corresponds to V. The last integral is bounded by our assumption. Denote the integrand of the first integral by $\ell(P', P, Q)$. It is continuous on $\Delta' \times V' \times \partial V$, where V' corresponds to $|\zeta| \leqq 1/2$. Since $h(P'; P, Q) = h(Q; P, P')$, $\ell(P', P, Q)$ is a harmonic function of Q on V for every fixed (P', P) on $\Delta' \times V'$. Hence there is $M' < \infty$ such that $|\ell(P', P, Q)|$ M' on $\Delta' \times V' \times V$. Thus I_2 is bounded on $\Delta' \times V'$.

The last integral $I_3(P', P)$ being bounded on $\Delta' \times V$ by Lemma 3, it is concluded that $h(P'; P, \mu)$ is bounded on $\Delta' \times V'$. This contradicts our assumption $\lim_{n\to\infty} |h(P'_n; P_n, \mu)| = \infty$.

Let us prove the latter half of our lemma. Choose $\delta > 0$ so that $|z| \leqq r_0 + \delta$ is still a closed parametric disk. Let W be its image. For $(P', P) \in W \times \Delta$ we write

$$k'(P', P, Q) = h(P'; P, Q) + \log |z(P') - z(P)|$$

and have

$$h(P'; P, \mu) - \mu(R) \log \frac{1}{|z(P')-z(P)|} = \int_{R-W} k'd\mu + \int_W k'd\mu.$$

Let W' be the image of $|z| \leqq r_0 + \delta/2$. By Lemma 3 $h(P'; P, Q)$ is continuous with respect to (P', P, Q) on $\partial W' \times \Delta \times (R - W)$ and hence so is k'. Let $|k'| < N$ there. For any fixed $(P, Q) \in \Delta \times (R - W)$, $k'(P', P, Q)$ is a harmonic function of P' on W' so that $|k'| \leqq N$ on $\Delta \times \Delta \times (R - W)$. Thus $\int_{R-W} k'd\mu$ is bounded on $\Delta \times \Delta$.

As to the integral on W we write it as

$$\int_W \{k'(P', P, Q) - \log |z(P') - z(Q)| + \log |z(P) - z(Q)|\}d\mu(Q)$$

$$+ \int_W \log |z(P') - z(Q)|d\mu(Q) - \int_W \log |z(P) - z(Q)|d\mu(Q).$$

By assumption the last two potentials are bounded on W. As above we see that

$$\ell'(P', P, Q) = k'(P', P, Q) - \log |z(P') - z(Q)| + \log |z(P) - z(Q)|$$

is bounded on $\partial W' \times \Delta \times \partial W$. Let $|\ell'| < N' < \infty$ there. Since ℓ' is a harmonic function of P' on W' for every fixed $(P, Q) \in \Delta \times \partial W$, $|\ell'| < N'$ on $\Delta \times \Delta \times \partial W$. By a similar reasoning we infer that $|\ell'| < N'$ on $\Delta \times \Delta \times W$. Thus $\int_W k'd\mu$ is bounded on $\Delta \times \Delta$. Accordingly $h(P'; P, \mu) + \mu(R)\log |z(P') - z(P)|$ is bounded on

$\Delta \times \Delta$. Our lemma is now completely proved.

The following result was orally suggested to the author in 1976 by Carleson for measures with continuous densities. A proof was given by Noguchi. We shall use Theorem 1 in the following proof.

Theorem 3. Suppose that locally the logarithmic potentials of μ and ν are both bounded, and that $\mu(R) = \nu(R)$. Then there exists a constant c independent of r, S and G such that

$$\left|\int_0^r \tilde{\mu}(G_t)dt - \int_0^r \tilde{\nu}(G_t)dt\right| < c.$$

Proof. By Theorem 1

$$\int_0^r \{\tilde{\mu}(G_t) - \tilde{\nu}(G_t)\}dt = \frac{1}{2\pi}\int_{\gamma_t - \gamma_0} \{h(f(\tilde{P}); P, \mu) - h(f(\tilde{P}); P, \nu)\}du_G^*(\tilde{P}).$$

On account of Lemma 7 we find a constant c' not depending on (P', P) such that

$$|h(P'; P, \mu) - h(P'; P, \nu)| < c'.$$

The required inequality follows immediately.

Remark. The condition on the boundedness of the logarithmic potential of μ is satisfied if, for instance, $d\mu$ is written locally as $\rho^2 dxdy$ with bounded density ρ^2.

In order to establish an inequality of the form $\mu(R)N(r, P) < \int_0^r \tilde{\mu}(G_t)dt + C$ we give

Lemma 8. Suppose that locally the logarithmic potential of μ is bounded. Then $\left|\int_{\gamma_0} h(f(\tilde{P}); P, \mu)du_G^*(\tilde{P})\right|$ is dominated by a finite constant which does not depend on P and G.

Proof. Let $\tilde{P}_0 \in \gamma_0$ and $P_0 = f(\tilde{P}_0)$. Define the conjugate u_G^* of u_G in a neighborhood of $\tilde{P}_0$ so that $u_G^*(\tilde{P}_0) = 0$. Since grad $u_G \neq 0$ on γ_0 and γ_0 does not contain any branch point of S as assumed in the beginning of §2, we may take $u_G + iu_G^*$ as a local parameter z not only around $\tilde{P}_0$ but also around P_0 on R. Let $|z| \leqq r_0 < 1$ be a closed local parametric disk, and $\tilde{\Delta}$ and Δ be the corresponding disks on S and R respectively. By Lemma 7 there exists $M < \infty$ such that $|h(P'; P, \mu)| < M$ on $\Delta' \times (R - \Delta)$ with the image Δ' of $|z| \leqq r_0/2$ on R, and

$$|h(P'; P, \mu) - \mu(R)\log \frac{1}{|z(P') - z(P)|}| < M$$

on $\Delta \times \Delta$. Denote the image of $|z| \leqq r_0/2$ on S by $\tilde{\Delta}'$. Then

$$\left| \int_{\gamma_0 \cap \tilde{\Delta}'} h(f(\tilde{P}); P, \mu) du_G^*(\tilde{P}) \right| \leqq M$$

if $P \in R - \Delta$ and

$$\left| \int_{\gamma_0 \cap \tilde{\Delta}'} h(f(\tilde{P}); P, \mu) du_G^*(\tilde{P}) \right| \leqq M + \mu(R) \int_{-\delta}^{\delta} \log \frac{1}{|u_G^* - z(P)|} du_G^*$$

$$\leqq M + \mu(R) \int_{-\delta}^{\delta} \log \frac{1}{|u_G^*|} du_G^*$$

if $P \in \Delta$ with some δ, $0 < \delta < 1$. Since γ_0 is covered by finitely many disks like $\tilde{\Delta}'$, our lemma is proved.

Next we give

Lemma 9. Fix $G_0 \supset S_0$ and let $\{G_k\}$ be a sequence of domains such that each $G_k \supset G_0$. Then there exists a positive harmonic function u in $G_0 - S_0$ which vanishes on γ_0 and to which some subsequence of $\{u_{G_k}\}$ converges.

Proof. Since each $u_{G_k} \geqq 0$ on $G_0 - S_0$, there is a subsequence of $\{u_{G_k}\}$ which tends to a harmonic function u or ∞ locally uniformly in $G_0 - S_0$. Denote the subsequence still by $\{u_{G_k}\}$, and assume that $u_{G_k} \to \infty$ on $G_0 - S_0$. Let $G^* = \{\tilde{P};\ u_{G_0}(\tilde{P}) < c_{G_0}/2\}$, and $\omega_{G^*-S_0}$ be the harmonic measure of $\partial G^* - \gamma_0$ with respect to $G^* - S_0$. Given any number $a > 0$, there exists k' such that $u_{G_{k'}} \geqq a\omega_{G^*-S_0}$ on $G_0 - S_0$, so that $\partial u_{G_{k'}}/\partial n \geqq a\partial\omega_{G^*-S_0}/\partial n$ on γ_0. Hence

$$1 = \int_{\gamma_0} \frac{\partial u_{G_{k'}}}{\partial n}\, ds \geqq a\int_{\gamma_0} \frac{\partial \omega_{G^*-S_0}}{\partial n}\, ds > 0.$$

This is impossible if a is large. Therefore, $u_{G_k} \to u$ as $k \to \infty$ on $G_0 - S_0$, in particular, uniformly on ∂G^*. We infer that $u = 0$ on γ_0, and that u is positive because $\int_{\gamma_0} \partial u/\partial n ds = 1$. Our lemma is now proved.

Theorem 4. Suppose that locally the logarithmic potential of μ is bounded. Then there is a constant C not depending on r and P such that

$$\mu(R)N(r, P) < \int_0^r \tilde{\mu}(G_t)dt + C.$$

If we fix G_0 and take only G which includes G_0, then we can choose C so that it does not depend on G.

Proof. Set $k(P', P) = h(P'; P, \mu) - \inf h(P'; P, \mu)$, where $m = \inf h(P'; P, \mu)$ on $R \times R$ is finite by Lemma 7. We have

$$\int_{\gamma_r-\gamma_0} h du_G^* = \int_{\gamma_r-\gamma_0} k du_G^*.$$

From Theorem 1 we derive

$$\begin{aligned}\mu(R)N(r, P) &= \int_0^r \tilde{\mu}(G_t)dt + \frac{1}{2\pi}\int_{\gamma_0} k\,du_G^* - \frac{1}{2\pi}\int_{\gamma_r} k\,du_G^* \\ &\leqq \int_0^r \tilde{\mu}(G_t)dt + \frac{1}{2\pi}\int_{\gamma_0} k\,du_G^* \\ &= \int_0^r \tilde{\mu}(G_t)dt + \frac{1}{2\pi}\int_{\gamma_0} h\,du_G^* - \frac{m}{2\pi}\int_{\gamma_0} du_G^* \\ &= \int_0^r \tilde{\mu}(G_t)dt + \frac{1}{2\pi}\int_{\gamma_0} h\,du_G^* - \frac{m}{2\pi}\,.\end{aligned}$$

By Lemma 8 $\int_{\gamma_0} h\,du_G^*$ is a bounded function of P on R so that its maximum M is finite. We conclude our theorem by taking $C = (M - m)/(2\pi)$.

To prove the latter part of the theorem, assume that every G contains G_0 and write $N_G(r, P)$ and C_G to show the dependence on G. Suppose that there exist $\{G_j\}$, each containing G_0, and $\{P_j\}$ on R such that $\int_{\gamma_0} k_j\,du_{G_j}^* \to \infty$, where $k_j = k(f(\tilde{P}), P_j)$. By Lemma 9 we may suppose that u_{G_j} converges to a harmonic function u in $G_0 - S_0$. Set $M' = \max_{\partial G^*} u$ with G^* in Lemma 9. We have

$$\frac{\partial u_{G_j}}{\partial n} \leqq (1 + M')\frac{\partial \omega_{G^*-S_0}}{\partial n} \leqq \frac{2(1+M')}{c_{G_0}}\frac{\partial u_{G_0}}{\partial n} \qquad \text{on } \gamma_0$$

for large j, and hence

$$\int_{\gamma_0} k_j \frac{\partial u_{G_j}}{\partial n}\,ds \leqq \frac{2(1+M')}{c_{G_0}}\int_{\gamma_0} k_j \frac{\partial u_{G_0}}{\partial n}\,ds \leqq \frac{4\pi(1+M')C_{G_0}}{c_{G_0}}\,.$$

This is a contradiction. It is now proved that there exists C not depending on G such that $\mu(R)N(r, P) < \int_0^r \tilde{\mu}(G_t)dt + C$.

§3. An identity

Suppose μ has density everywhere on R. Thus $d\mu = \rho_z^2 dxdy$ locally. We call $\rho_z|dz|$ a conformal metric on R. If ζ is another local parameter, then $\rho_z|dz| = \rho_\zeta|d\zeta|$. We assume in this section that $\rho_z \in C^2$ and ρ_z is positive everywhere on R and that $\iint_R \rho_z^2 dxdy = 1$. Cover R by open disks so that every point of R belongs to only finitely many disks. Let $|w| < 1$ correspond to such a disk and assume that $|w| < 1 + \varepsilon$ corresponds to an open disk too. Let $\rho_w = \rho_w(w)$ be equal to 1 on $|w| < 1$, equal to 0 outside of $|w| < 1 + \varepsilon/2$ and non-negative and of class C^2 on $|w| < 1 + \varepsilon$. Regard $\rho_w|dw|$ as a kind of conformal metric. We form such a metric to every disk. The sum is a positive conformal metric of class C^2 on R.

We form the Gaussian curvature

$$K = -\frac{\Delta \log \rho_z}{\rho_z^2} = -\frac{\Delta_z \log \rho_z}{\rho_z^2}.$$

Let us see that K is invariant under $z \to \zeta$. Actually, we have

$$\Delta_z \log \rho_z = \Delta_\zeta\{\log(\rho_\zeta|\tfrac{d\zeta}{dz}|)\}|\tfrac{d\zeta}{dz}|^2 = (\Delta_\zeta \log \rho_\zeta)\frac{\rho_z^2}{\rho_\zeta^2},$$

which yields $\rho_z^{-2}\Delta_z \log \rho_z = \rho_\zeta^{-2}\Delta_\zeta \log \rho_\zeta$.

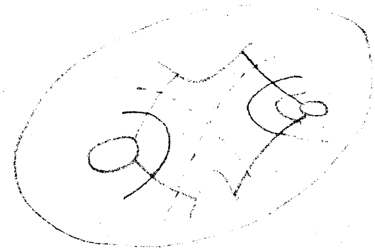
Very often Gauss-Bonnet's formula is derived and applied to obtain an identity which will follow. We shall choose a different way. According to [7; p.251] the following relation is basically due to Poincaré and Bendixson. In proving it we shall follow [7]. As to other proofs see [10; p.35] and [11; §1.3].

For a harmonic function ω in D we call a point at which grad $\omega = 0$ a critical point. Let $P_1, \ldots, P_m$ be the critical points, and n_i be the multiplicity of grad ω at P_i. Set $n(\text{grad } \omega, D) = \sum_{i=1}^{m} n_i$.

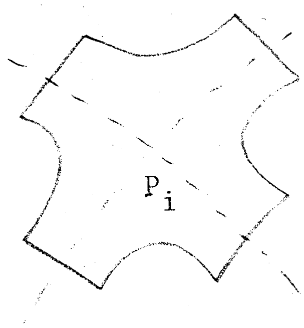
Lemma 10. Let D be a relatively compact domain on a Riemann surface bounded by k $(2 \leqq k < \infty)$ analytic closed curves. Let $E \neq \partial D$ be a non-empty set of closed curves on ∂D and ω be a harmonic function on D which is equal to 0 on E and to a positive constant on $\partial D - E$. Then

$$\chi(D) = n(\text{grad } \omega, D),$$

where $\chi(D)$ is the characteristic of D.



Proof. We divide D into m (curvilinear) polygons and α_2 (curvilinear) rectangles such that each side lies on a level curve or on an orthogonal trajectory and each polygon surrounds just one P_i as in the figure. One fourth of the number of its corners is equal to $n_i + 1$. Denote by D' the complement of m polygons. We assume that rectangles form a mesh. Let $\alpha_{0,2}$ be the number of corners of D' each of which belongs just to two rectangles; they lie on ∂D. Let $\alpha_{0,3}$ be the number of corners on polygons and $\alpha_{0,4}$ be the number of corners lying in the interior of D'. Then the number α_0 of corners is equal to $\alpha_{0,2} + \alpha_{0,3} + \alpha_{0,4}$. The number α_1 of edges is equal to



$$\frac{1}{2}(3\alpha_{0,2} + 4\alpha_{0,3} + 4\alpha_{0,4})$$

and

$$\alpha_2 = \frac{1}{4}(2\alpha_{0,2} + 3\alpha_{0,3} + 4\alpha_{0,4}).$$

Hence

$$\chi(D') = -\alpha_0 + \alpha_1 - \alpha_2 = \frac{1}{4}\alpha_{0,3} = \sum n_i + m = n(\text{grad } \omega, D) + m.$$

Since $\chi(D) = \chi(D') - m$, $\chi(D) = n(\text{grad } \omega, D)$. Our lemma is now proved.

Let $|z| \leqq r_0$ correspond to a closed disk on R. Let γ be a smooth curve on $0 < |z| \leqq r_0$ and form

$$d\tau = d\theta_z + \frac{\partial \log \rho_z}{\partial n_z} ds_z \qquad \text{along } \gamma,$$

where θ_z is the angle between the tangent and the x-axis. If z is transformed to ζ, then $\log \rho_\zeta(\zeta) - \log \rho_z(z) = \log(|dz/d\zeta|)$ and

$$\frac{\partial}{\partial n_\zeta} \log \left|\frac{dz}{d\zeta}\right| \cdot ds_\zeta = d \arg \frac{dz}{d\zeta} = d\theta_z - d\theta_\zeta.$$

Hence

$$d\theta_z + \frac{\partial \log \rho_z}{\partial n_z} ds_z = d\theta_\zeta + \frac{\partial \log \rho_\zeta}{\partial n_\zeta} ds_\zeta. \tag{4}$$

Therefore $d\tau$ is invariant.

We define $\tilde{K}$ on S by $\tilde{K}(\tilde{P}) = K(f(\tilde{P}))$. We prove

Lemma 11. Let F be a relatively compact subdomain of S with boundary which consists of finitely many analytic arcs and which does not contain any branch point of S. Then

$$\frac{1}{2\pi}\int_F \tilde{K} d\tilde{\mu} = b(F) - \chi(F) - \frac{1}{2\pi}\int_{\partial F} d\tau - \frac{1}{2\pi}\sum \tau_i,$$

where b(F) is the sum of the orders of the branch points of F and $\tau_1, \tau_2, \ldots$ are the changes of angles at the corners of ∂F.

Proof. Let F_0 be a closed disk contained in F such that F_0 does not contain branch points of S. Let ω be the harmonic measure of ∂F with respect to $F - F_0$. Let $\Delta_1, \ldots, \Delta_k$ be mutually disjoint closed disks with centers at the corners of ∂F and $\Delta_1', \ldots, \Delta_\ell'$ be mutually disjoint closed disks in $F - \Delta_1 - \cdots - \Delta_k$ around the cirtical points of ω and around the branch points. At every point $\tilde{P}$ of $F - F_0$ at which grad $\omega \neq 0$, $\omega + i\omega^*$ may be taken as a local parameter. Denote $\rho_{\omega+i\omega^*}$ simply by ρ_ω. By Green's formula we have

$$\int_{\partial(F-F_0-\cup\Delta_i-\cup\Delta_j')} \frac{\partial \log \rho_\omega}{\partial n_z} ds_z = \iint_{F-F_0-\cup_i\Delta_i-\cup_j\Delta_j'} \Delta \log \rho_\omega dxdy$$

$$= \iint_{F-F_0-\cup_i\Delta_i-\cup_j\Delta_j'} \Delta \log \rho_z dxdy$$

because $\rho_z = \rho_\omega|\mathrm{grad}_z\omega|$ and $\log |\mathrm{grad}_z\omega|$ is harmonic. Let us compute the limit of $\int_{\partial\Delta_j'} (\partial \log \rho_\omega/\partial n)ds$ as Δ_j' shrinks to its center $\tilde{P}_0$. We shall treat the case where $\tilde{P}_0$ may be at the same time a critical and branch point. Let $n(\tilde{P}_0)$ be the multiplicity of f at $\tilde{P}_0$ and p be the multiplicity of grad ω at $\tilde{P}_0$. Write ds_ω, etc. for $ds_{\omega+i\omega^*}$, etc. By (4) we have

$$\frac{1}{2\pi}\int_{\partial\Delta_j'} \frac{\partial \log \rho_\omega}{\partial n_\omega} ds_\omega$$

$$= \frac{1}{2\pi}\int_{f(\partial\Delta_j')} \frac{\partial \log \rho_z}{\partial n_z} ds_z + \frac{1}{2\pi}\int_{f(\partial\Delta_j')} d\theta_z - \frac{1}{2\pi}\int_{\partial\Delta_j'} d\theta_\omega$$

$$\to n(\tilde{P}_0) - p - 1 \qquad \text{as } \Delta_j' \to \tilde{P}_0.$$

As to the integral along $\partial\Delta_i \cap F$

$$\frac{1}{2\pi}\int_{\partial\Delta_i\cap F} \frac{\partial \log \rho_\omega}{\partial n_\omega} ds_\omega$$

$$= \frac{1}{2\pi}\int_{f(\partial\Delta_i\cap F)} \frac{\partial \log \rho_z}{\partial n_z} ds_z + \frac{1}{2\pi}\int_{f(\partial\Delta_i\cap F)} d\theta_z - \frac{1}{2\pi}\int_{\partial\Delta_i\cap F} d\theta_\omega$$

$$\to -\frac{1}{2\pi}(\pi - \tau_i) + \frac{\pi}{2\pi} = \frac{1}{2\pi}\tau_i.$$

Accordingly

$$\frac{1}{2\pi}\iint_{F-F_0} \Delta \log \rho_z dxdy$$

$$= \frac{1}{2\pi}\int_{\partial(F-F_0)} \frac{\partial \log \rho_\omega}{\partial n_z} ds_z - b(F-F_0) + \frac{1}{2\pi}\sum \tau_i + n(\text{grad}\ \omega, F-F_0).$$

By Lemma 10 we obtain

$$\frac{1}{2\pi}\int_{F-F_0} \tilde{K}d\tilde{\mu} = b(F-F_0) - \chi(F-F_0) - \frac{1}{2\pi}\int_{\partial(F-F_0)} d\tau - \frac{1}{2\pi}\sum \tau_i.$$

We note that

$$\frac{1}{2\pi}\int_{F_0} \tilde{K}d\tilde{\mu} = -\frac{1}{2\pi}\int_{\partial F_0} \frac{\partial \log \rho_z}{\partial n} ds = -\frac{1}{2\pi}\int_{\partial F_0} d\tau + 1$$

and obtain

$$\frac{1}{2\pi}\int_{F} \tilde{K}d\tilde{\mu} = b(F) - \chi(F) - \frac{1}{2\pi}\int_{\partial F} d\tau - \frac{1}{2\pi}\sum \tau_i.$$

If $F = S = R$ then we have

Corollary.

(5) $$\frac{1}{2\pi}\int_{R} Kd\mu = -\chi(R).$$

We set

$$T(r) = \int_0^r \tilde{\mu}(G_t)dt, \quad E(r) = \int_0^r \chi(G_t)dt, \quad B(r) = \int_0^r b(G_t)dt.$$

The main result in this section is

Theorem 5.

$$B(r) - E(r) + \chi(R)T(r) = \frac{1}{2\pi}\int_{\gamma_r - \gamma_0} (\log \rho_u - U)\,du_G^*,$$

where

$$U(\tilde{P}) = \frac{1}{2\pi}\int_R h(f(\tilde{P});\, P,\, \mu)K(P)\,d\mu(P).$$

Proof. By Lemma 11 we have

$$\frac{1}{2\pi}\int_0^r\int_{G_t} \tilde{K}d\tilde{\mu}dt = \int_0^r b(G_t)dt - \int_0^r \chi(G_t)dt - \frac{1}{2\pi}\int_0^r dt\int_{\tau_t} \frac{\partial \log \rho_u}{\partial t}\, du_G^*$$

$$= B(r) - E(r) - \frac{1}{2\pi}\int_{\gamma_r - \gamma_0} \log \rho_u du_G^*.$$

Integrating (3) in Theorem 1 with respect to $Kd\mu$, we derive

$$T(r)\int_R Kd\mu - \int_{\gamma_r - \gamma_0} Udu_G^* = \int_0^r dt\int_R n(t,\, P)K(P)d\mu(P)$$

$$= \int_0^r\int_{G_t} \tilde{K}d\tilde{\mu}dt.$$

We use (5) and obtain

$$B(r) - E(r) + \chi(R)T(r) = \frac{1}{2\pi}\int_{\gamma_r - \gamma_0} (\log \rho_u - U)\,du_G^*.$$

Our theorem is thus proved.

Remark. If K is constant, then the right hand side of the identity in the theorem reduces to $(2\pi)^{-1}\int_{\gamma_r - \gamma_0} \log \rho_u du_G^*$.

We shall extend the identity in Theorem 5. The method is due to Ahlfors [1]. Take $P_1, \ldots, P_q$ on R. Let $|z_\nu| < 1$ correspond to a disk with center P_ν. We assume that they are mutually disjoint. Let $\rho_z |dz|$ be a positive conformal metric on $R - \{P_1, \ldots, P_q\}$ such that $\iint_R \rho_z^2 dxdy = 1$ and

$$\text{(6)} \qquad \rho_{z_\nu} = \frac{1}{|z_\nu|}\left(\log\frac{1}{|z_\nu|}\right)^{-2} \qquad \text{on } |z_\nu| < r_\nu,$$

where $r_1, \ldots, r_q$ are chosen so that each $r_\nu < e^{-1}$. We note that $\Delta \log \rho_{z_\nu} = 2\,|z_\nu|^{-2}(\log(1/|z_\nu|))^{-2}$ and hence that $\int_R |K| d\mu < \infty$.

We prove

Lemma 12. With ρ_{z_ν} in (6) we have

$$\frac{1}{2\pi}\iint_{|z_\nu|<r_0}\left(\log\frac{1}{|z-z_\nu|}\right)\Delta \log \rho_{z_\nu}\, dx_\nu dy_\nu$$

$$= 2 \log\log\frac{1}{|z|} - 2\log\log\frac{1}{r_0} + 2 \qquad \text{for } |z| < r_0,$$

where $z_\nu = x_\nu + iy_\nu$ and $r_0 < e^{-1}$.

Proof. Denote the potential by $V(z)$. With polar coordinates we have $\Delta \log \rho_{z_\nu} = 2r^{-2}(\log(1/r))^2$ as above. Hence

$$V(z) = \frac{1}{\pi}\int_0^{r_0}\left(\int_0^{2\pi}\log\frac{1}{|z-re^{i\theta}|}\,d\theta\right)\frac{1}{r(\log(1/r))^2}\,dr$$

$$= 2\log\frac{1}{|z|}\int_0^{|z|}\frac{dr}{r(\log 1/r)^2} + 2\int_{|z|}^{r_0}\frac{dr}{r\log 1/r}$$

$$= 2 + 2\log\log\frac{1}{|z|} - 2\log\log\frac{1}{r_0}\,.$$

Let G be a subdomain of S as in §2 and $u = u_G$ be as there. We shall prove

Lemma 13. $\int_{\gamma_t}(\log \rho_u - U)du^*$ is a continuous function of t, where

$$U(\tilde{P}) = \frac{1}{2\pi}\int_R h(f(\tilde{P});\, P,\, \mu)K(P)d\mu(P).$$

Proof. Let $\tilde{P}_0 \in \gamma_r$. It suffices to consider the continuity at $t = r$. As in the proof of Lemma 4 we set $w^p = F = u + iu^* - u(\tilde{P}_0)$. Let $z = x + iy$ be a local parameter on R such that $z = 0$ corresponds to $f(\tilde{P}_0)$, and write

$$z(f(\tilde{P}(w))) = z(w) = w^q g(w) \qquad (q \geqq 1)$$

with $g(0) \neq 0$. Then

$$\rho_w = \left|\frac{dz}{dw}\right|\rho_z = |w|^{q-1}|qg(w) + wg'(w)|\rho_z.$$

The identity $\rho_u|d(u + iu^*)| = \rho_w|dw|$ yields

$$\begin{aligned}\log \rho_u &= (1 - p)\log |w| + \log \rho_w - \log p \\ &= (q - p)\log |w| + \log \rho_z + G(w),\end{aligned}$$

where G is a continuous function.

Suppose $f(\tilde{P}_0)$ coincides with none of $P_1, \ldots, P_q$. Let V be a closed disk on R which contains none of $P_1, \ldots, P_q$ and which corresponds to $|z| \leqq r_0$. Denote by V' the image of $|z| \leqq r_0/2$. For $P' \in V'$ we have

$$\begin{aligned}\int_R h(P'; P, \mu)K(P)d\mu(P) &= \int_{R-V} h(P'; P, \mu)K(P)d\mu(P) \\ &+ \int_V \left\{h(P'; P, \mu) - \log \frac{1}{|z(P')-z(P)|}\right\}K(P)d\mu(P) \\ &+ \iint_{|z|\leqq r_0} \log \frac{1}{|z(P')-z|}\, \Delta \log \rho_z \, dxdy.\end{aligned}$$

Since $\Delta \log \rho_z$ is continuous on $|z| \leqq r_0$, the last integral is bounded on V'. The first two integrals are bounded on V' on account of

Lemma 7. If $f(\tilde{P}_0)$ coincides with some P_ν, then let $z = z_\nu$. This choice of local parameter gives

$$\log \rho_z = \log \rho_{z_\nu} = -\log |z_\nu| - 2 \log \log \frac{1}{|z_\nu|}.$$

By Lemmas 7 and 12 we infer that

$$\log \rho_z - U = -\log |z_\nu| + \phi(z_\nu)$$

in $|z_\nu| < 1$ with a bounded function $\phi(z_\nu)$, and hence

$$\log \rho_u - U = (q - p)\log |w| - q \log |w| - \log |g| + \phi(z_\nu) + G(w)$$

$$= -p \log |w| + \text{a bounded function.}$$

The proof of our lemma is completed as in the proof of Lemma 4.

We shall establish

Theorem 6.

$$\sum_{\nu=1}^{q} (T(r) - N(r, P_\nu)) + B(r) - E(r) + \chi(R)T(r)$$

(7)

$$= \frac{1}{2\pi}\int_{\gamma_r - \gamma_0} (\log \rho_u - U)du^*.$$

Proof. Around each P_ν we draw a small closed disk $\Delta_\nu(\varepsilon)$ corresponding to $|z_\nu| \leqq \varepsilon < r_\nu$. Suppose the projection of $G_t \cup \gamma_t - G_t$, contains none of $P_1, \ldots, P_q$. Choose ε so that $f(\gamma_t)$ is disjoint from $\cup_\nu \partial\Delta_\nu(\varepsilon)$ and set

$$G_t(\varepsilon) = G_t - f^{-1}(\cup_\nu \Delta_\nu(\varepsilon)).$$

By Lemma 11

$$\frac{1}{2\pi}\int_{G_t(\varepsilon)} \tilde{K}d\tilde{\mu} = b(G_t(\varepsilon)) - \chi(G_t(\varepsilon)) - \frac{1}{2\pi}\int_{f^{-1}(\cup_\nu \partial\Delta_\nu(\varepsilon))\cup\partial G_t} d\tau.$$

If $\tilde{P}$ is not a branch point and $f(\tilde{P}) = P_\nu$, then

$$\frac{1}{2\pi}\int_{f^{-1}(\partial\Delta_\nu(\varepsilon))} d\tau = \frac{1}{2\pi}\int_{\partial\Delta_\nu(\varepsilon)} \frac{\partial \log \rho_z}{\partial n} ds + \frac{1}{2\pi}\int_{\partial\Delta_\nu(\varepsilon)} d\theta_z$$

$$= -\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{\varepsilon} + \frac{2}{\varepsilon}\frac{1}{\log \varepsilon}\right)\varepsilon d\theta + 1$$

$$\to 0 \qquad\qquad \text{as } \varepsilon = |z_\nu| \to 0.$$

The same is true even if $\tilde{P}$ is a branch point and $f(\tilde{P}) = P_\nu$. Let $b'(G_t)$ be the sum of the orders of the branch points of G_t lying above $P_1, \ldots, P_q$. Then $\sum_\nu n(t, P_\nu) - b'(G_t)$ is equal to the number of branch points of G_t lying above $P_1, \ldots, P_q$. We have

$$b(G_t(\varepsilon)) - \chi(G_t(\varepsilon)) = b(G_t) - b'(G_t) - \chi(G_t) - \{\sum_\nu n(t, P_\nu) - b'(G_t)\}$$

$$= b(G_t) - \chi(G_t) - \sum_\nu n(t, P_\nu).$$

Since $\int_{\Delta_\nu(\varepsilon)} Kd\mu \to 0$ as $\varepsilon \to 0$,

$$\frac{1}{2\pi}\int_{G_t} \tilde{K}d\tilde{\mu} = b(G_t) - \chi(G_t) - \sum_\nu n(t, P_\nu) - \frac{1}{2\pi}\int_{\gamma_t} \frac{\partial \log \rho_u}{\partial t} du^*.$$

Similarly we derive

$$\frac{1}{2\pi}\int_R Kd\mu = -\chi(R) - q.$$

We obtain

$$\frac{1}{2\pi}\int_{t'}^{t}\int_{G_{t''}} \tilde{K}d\tilde{\mu}dt'' = B(t) - B(t') - E(t) + E(t') - \frac{1}{2\pi}\int_{\gamma_t - \gamma_{t'}} \log \rho_u du^*.$$

Integrating (3) with respect to $Kd\mu$, we have

$$(T(t) - T(t'))\int_R K d\mu - \int_{\gamma_t - \gamma_{t'}} U du^*$$

$$= -2\pi(T(t) - T(t'))(\chi(R) + q) - \int_{\gamma_t - \gamma_{t'}} U du^* = \int_{t'}^{t}\int_{G_{t''}} \tilde{K} d\tilde{\mu} dt'',$$

and obtain

$$B(t) - B(t') - E(t) + E(t') + (\chi(R) + q)(T(t) - T(t'))$$

$$- \sum_{\nu=1}^{q} N(t, P_\nu) + \sum_{\nu=1}^{q} N(t', P_\nu)$$

$$= \frac{1}{2\pi}\int_{\gamma_t - \gamma_{t'}} (\log \rho_u - U) du^*.$$

Since $\int_{\gamma_t} (\log \rho_u - U) du^*$ is a continuous function of t by Lemma 13, our theorem is concluded.

§4. Second main theorem

We begin with

Lemma 14. Let $\rho_z |dz|$ be a positive conformal metric which may have singularities like (6). Fix $G_0 \supset S_0$. Then there exists a constant c such that

$$\int_{\gamma_0} \log^- \rho_{u_G} du_G^* \leqq c < \infty$$

for all $G \supset G_0$.

Proof. Suppose there exists $\{G_k\}$ such that each $G_k \supset G_0$ and $\int_{\gamma_0} \log^- \rho_{u_{G_k}} du_{G_k}^* \to \infty$ as $k \to \infty$. For simplicity write u_k for u_{G_k}. By Lemma 9 there exists a positive harmonic function u which vanishes on γ_0 and to which a subsequence of $\{u_k\}$ converges. We write still $\{u_k\}$ for it. Set

$$H_k = u_k + iu_k^* \qquad \text{and} \qquad H = u + iu^*.$$

At every point of γ_0 we may take H as a local parameter. As $k \to \infty$ $|dH_k/dH| \to 1$ on γ_0 and hence

$$-\log \rho_{u_k} = -\log \rho_u + \log \left|\frac{dH_k}{dH}\right|$$

is bounded from above on γ_0 for $k = 1, 2, \ldots.$ This contradicts the fact that $\int_{\gamma_0} \log^- \rho_{u_k} du_k^* \to \infty$ as $k \to \infty$. Our lemma is thus proved.

Lemma 15. Let λ be a non-negative measure in a measure space, and B be a measurable set with $\lambda(B) > 0$. If ϕ is a non-negative and λ-integrable function on B, then

$$\frac{1}{\lambda(B)}\int_B (\log \phi) d\lambda \leqq \log \left\{\frac{1}{\lambda(B)}\int_B \phi d\lambda\right\}.$$

Proof. We may assume that $\int_B \phi d\lambda > 0$. Set $c = (1/\lambda(B))\int_B \phi d\lambda$, and $\psi = \phi - c$. Then $\int_B \psi d\lambda = 0$ and $1 + \psi/c \geqq 0$. For every $t \geqq -1$, $\log (1 + t) \leqq t$. Hence $\log (1 + \psi/c) \leqq \psi/c$, and

$$\frac{1}{\lambda(B)}\int_B (\log \phi) d\lambda = \frac{1}{\lambda(B)} \left\{\int_B \log c d\lambda + \int_B \log \left(1 + \frac{\psi}{c}\right) d\lambda\right\}$$

$$\leqq \log c + \frac{1}{\lambda(B)}\int_B \frac{\psi}{c} d\lambda = \log c = \log \left\{\frac{1}{\lambda(B)}\int_B \phi d\lambda\right\}.$$

Our lemma is thus proved.

Lemma 16. Let $\rho_z|dz|$ be the conformal metric given before Lemma 12. Then

$$\left|\int_{\gamma_r} U(\tilde{P}) du^*(\tilde{P})\right| \leqq 2q \log (T(r)+\text{const.}) + \text{const.},$$

where constants do not depend on r and G.

Proof. Let D_ν be the open disk corresponding to $|z_\nu| < \dot{r}_\nu$. For $P' \in D_\nu$ we have

$$\int_R h(P';\, P,\, \mu)K(P)d\mu(P) = \int_{R-D} h(P';\, P,\, \mu)K(P)d\mu(P)$$

$$+ \int_{D_\nu}\left\{h(P';\, P,\, \mu) - \log \frac{1}{|z_\nu(P')-z_\nu(P)|}\right\}K(P)d\mu(P)$$

$$+ \int_{D_\nu} \log \frac{1}{|z_\nu(P')-z_\nu(P)|} K(P)d\mu(P).$$

By Lemma 7 the first two integrals are bounded, and by Lemma 12

$$\frac{1}{2\pi}\int_{D_\nu} \log \frac{1}{|z_\nu(P')-z_\nu(P)|} K(P)d\mu(P) = -\,2 \log \log \frac{1}{|z_\nu(P')|} + 2.$$

Denote by c_ν the part of γ_r whose projection is contained in D_ν. We observe that

$$\int_{\gamma_r} |U(\tilde{P})|du^*(\tilde{P}) \leqq 2 \sum_{\nu=1}^{q} \int_{c_\nu} \log \log \frac{1}{|z_\nu(f(\tilde{P}))|} du^*(P) + \text{const.}$$

$$\leqq 2 \sum_{\nu=1}^{q} |c_\nu| \log\left(|c_\nu|^{-1}\int_{c_\nu} \log \frac{1}{|z_\nu(f(\tilde{P}))|} du^*(\tilde{P})\right) + \text{const.}$$

$$\leqq 2qe^{-1} + 2 \sum_{\nu=1}^{q} \log \int_{\gamma_r} \log \frac{1}{|z_\nu(f(\tilde{P}))|} du^*(\tilde{P}) + \text{const.}$$

by Lemma 15, where $|c_\nu|$ is the u*-measure of the set c_ν. By the aid of Lemma 7 and Theorem 1 we infer that

$$\log \int_{\gamma_r} \log \frac{1}{|z_\nu(f(\tilde{P}))|} du^*(\tilde{P})$$

$$\leqq \log\left|\int_{\gamma_r} h(f(\tilde{P});\, P_\nu,\, \mu)du^*(\tilde{P})\right| + \text{const.}$$

$$\leqq \log \left\{2\pi|T(r) - N(r, P_\nu)| + \left|\int_{\gamma_0} h(f(\tilde{P}); P_\nu, \mu)du^*(\tilde{P})\right|\right\} + \text{const.}$$

This is

$$\leqq \log (T(r) + \text{const.}) + \text{const.}$$

on account of Theorem 4 and Lemma 8. Hence

$$\left|\int_{\gamma_r} U(\tilde{P})du^*(\tilde{P})\right| \leqq 2q \log (T(r) + \text{const.}) + \text{const.}$$

This proves our lemma.

Now Theorem 6, Lemmas 14 and 16 give

Theorem 7. Take $P_1, \ldots, P_q$ on R, and $\rho_z|dz|$ as before Lemma 12. Then, for any G containing a fixed $G_0 \supset S_0$,

$$\sum_{\nu=1}^{q} (T(r) - N(r, P_\nu)) + B(r) - E(r) + \chi(R)T(r) - \frac{1}{2\pi}\int_{\gamma_r} \log \rho_u du^*$$

$$\leqq 2q \log (T(r) + \text{const.}) + \text{const.}$$

Set $w(r) = \int_{\gamma_r} \log \rho_u du^*$. By Lemma 15

$$2w(t) \leqq \log\left(\int_{\gamma_t} \rho_u^2 du^*\right)$$

and hence

$$\int_0^r e^{2w(t)}dt \leqq \int_0^r\int_{\gamma_t} \rho_u^2 du^* dt = \tilde{\mu}(G_r)$$

so that

(8)
$$\int_0^r dt\int_0^t e^{2w(s)}ds \leqq T(r).$$

By means of Theorem 7 we have

Theorem 8. Take $P_1, \ldots, P_q$ on R, and $\rho_z|dz|$ as before Lemma 12. Then, for any G containing a fixed $G_0 \supset S_0$,

$$\sum_{\nu=1}^{q}(T(r) - N(r, P_\nu)) + B(r) - E(r) + \chi(R)T(r) - (2\pi)^{-1}w(r)$$
$$\leqq 2q \log (T(r) + \text{const.}) + \text{const.},$$

where $w(r)$ satisfies (8).

Remark. On account of Theorem 3 this inequality is valid for any μ whose logarithmic potential is locally bounded on R.

To evaluate w from above we give

Lemma 17. Fix G_0 and G_0' so that $s_0 \subset G_0'$ and $G_0' \cup \partial G_0' \subset G_0$. Then there exists a, $0 < a < 1$, such that $\{\tilde{P}; u_G(\tilde{P}) \leqq a\}$ is contained in G_0' for every $G \subset G_0$.

Proof. Suppose there exists $\{G_k\}$ such that each G_k contains G_0 and inf u_{G_k} on $\partial G_0'$ tends to 0 as $k \to \infty$. By Lemma 9 we may suppose that $\{u_{G_k}\}$ converges to a positive harmonic function u locally uniformly in $G_0 - S_0$. It follows that inf u on $\partial G_0'$ is zero. This is impossible because u is positive on $G_0 - S_0$. Thus there exists a', $0 < a' < 1$, such that $u_G > a'$ on $\partial G_0'$ for all $G \supset G_0$. By the minimum principle $u_G > a$ on $G - G_0'$ for all $G \supset G_0$, where $a = \min(a', c_{G_0}/2)$. Therefore, $\{\tilde{P} \in G; u_G(\tilde{P}) \leqq a\}$ is contained in G_0'.

Next we verify

Lemma 18. Let ψ be a continuous increasing function on $[r_0, r^*]$, and $\Psi(t)$ be a positive continuous function defined on $[t_0, \infty)$ with $\int dr/\Psi(r) < \infty$. If $\psi(r_0) \geqq t_0$, then $r\psi'(r) \leqq \Psi(\psi(r))$ on $[r_0, r^*)$ except a measurable subset I with $\int_I d \log r < \int dr/\Psi(r)$.

Proof. Let I be the subset of $[r_0, r^*)$ on which $r\psi'(r) > \Psi(\psi(r))$. It is certainly a measurable set, and

$$\int_I d \log r \leqq \int_I \frac{\psi'(r)}{\Psi(\psi(r))} dr \leqq \int_I \frac{d\psi(r)}{\Psi(\psi(r))} \leqq \int \frac{dt}{\Psi(t)} .$$

Our lemma is thus proved.

We shall establish the second main theorem.

Theorem 9. Take $P_1, \ldots, P_q$ arbitrarily on R, and $\rho_z|dz|$ as before Lemma 12. Suppose $c_{G_0} > 1$ for $G_0 \supset S_0$. Then

$$\sum_{\nu=1}^{q} (T_G(r) - N_G(r, P_\nu)) + B_G(r) \tag{9}$$
$$\leqq E_G(r) - \chi(R)T_G(r) + b \log T_G(r) + b'$$

for any $G \supset G_0$ and $r \in [1, c_G) - I$, where b and b' are finite constants independent of G and r, and where I is a measurable subset of $[1, c_G)$ such that $\int_I d \log t$ is bounded above by a finite constant independent of G and r.

Proof. We shall use Theorem 8. By Theorems 4 and 7 we have

$$\frac{1}{2\pi}\int_{\gamma_r} \log \rho_{u_G} du_G^* \geqq - E_G(r) - |\chi(R)|T_G(r) + \text{const.}$$

Lemma 17 implies that there is a, $0 < a < 1$, such that $\{\tilde{P}; u_G(\tilde{P}) \leqq r\} \subset G_0$ for all $G \supset G_0$ and $r \in [0, a]$. It follows that $E_G(r) \leqq a|\chi(G_0)|$ and $T_G(r) \leqq a\tilde{\mu}(G_0)$ if $G \supset G_0$ and $0 < r \leqq a$. Hence $\int_{\gamma_r} \log \rho_{u_G} du_G^*$ is bounded below so that there is a finite constant c' such that $w(r) > c'$ if $0 < r \leqq a$.

We apply Lemma 18 to $\psi(r) = \int_0^r e^{2w(s)}ds$ on $[1, c_G)$ and $\Psi(r) = r^\beta$ with $\beta > 1$ on $[t_0, \infty)$, where $t_0 = \int_0^a e^{2w(s)}ds$. We obtain

$$e^{2w(t)} \leqq \left(\int_0^t e^{2w(s)}ds\right)^\beta \qquad \text{on } [1, c_G) - I,$$

where I is a measurable set satisfying

$$\int_I d\log r < \frac{1}{\beta-1}\left(\int_0^a e^{2w(s)}ds\right)^{1-\beta} \leqq \frac{1}{\beta-1}\,\frac{1}{(ae^{2c'})^{\beta-1}} < \infty.$$

Applying Lemma 18 next to $\psi(r) = \int_0^r dt \int_0^t e^{2w(s)}ds$ we see that

$$\int_0^r e^{2w(s)}ds \leqq \left(\int_0^r dt \int_0^t e^{2w(s)}ds\right)^\beta \qquad \text{on } [1, c_G) - I',$$

where I' is a measurable set satisfying

$$\int_{I'} d\log r < \frac{1}{\beta-1}\left(\frac{2}{a^2e^{2c'}}\right)^{\beta-1} < \infty.$$

For $r \in [1, c_G) - I - I'$ we obtain

$$w(r) \leqq \frac{\beta^2}{2}\log T_G(r).$$

By Theorem 8 and the relation $\gamma\tilde{\mu}(S_0) \leqq T(r)$ we have

$$\sum_{\nu=1}^{q}(T_G(r) - N_G(r, P_\nu)) + B_G(r)$$

$$\leqq E_G(r) - \chi(R)T_G(r) + 2q\log T_G(r) + \frac{\beta^2}{4\pi}\log T_G(r) + c^*,$$

where c* is a constant independent of G. Our theorem is now proved.

§5. Defect relation

We shall prove

Theorem 10. Let S be a parabolic open Riemann surface, and $\{G_n\}$ be any exhaustion. Then there exists $\{r_n\}$ tending to ∞ such that $0 < r_n < c_{G_n}$ for each n and

(10) $$\sum_{\nu=1}^{q} \gamma(P_\nu) + b \leqq \xi - \chi(R),$$

where

$$\gamma(P_\nu) = 1 - \limsup_{n\to\infty} \frac{N_{G_n}(r_n, P_\nu)}{T_{G_n}(r_n)}, \quad b = \limsup_{n\to\infty} \frac{B_{G_n}(r_n)}{T_{G_n}(r_n)},$$

$$\xi = \limsup_{n\to\infty} \frac{E_{G_n}(r_n)}{T_{G_n}(r_n)}.$$

Proof. We note that S is parabolic if and only if $c_{G_n} \uparrow \infty$ as $n \to \infty$. Hence we may assume that all $c_{G_n} > 1$. For each n we choose $r_n > c_{G_n}/2$ satisfying (9) with $G = G_n$. As $n \to \infty$ $r_n \to \infty$ and $T_{G_n}(r_n) \to \infty$ so that (9) yields (10).

Remark 1. The existence of the following function p on any parabolic open Riemann surface is known (cf. Chap. IV of [11]):

(i) p is harmonic outside a point P_0 of R,

(ii) p has a logarithmic singularity at P_0, i.e.,

$$p(P(z)) - \log |z|$$

is harmonic in a neighborhood of $z = 0$, where z is a local parameter around P_0 and $z = 0$ corresponds to P_0,

(iii) $p(P) \to \infty$ as P tends to the ideal boundary of R.

Set $G_r = \{P \in R;\ p(P) < r\}$ and

$$\gamma^*(P_\nu) = 1 - \limsup_{r\to\infty} \frac{\int_{r_0}^{r} n(t, P_\nu)dt}{\int_{r_0}^{r} \tilde{\mu}(G_t)dt}, \quad b^* = \limsup_{r\to\infty} \frac{\int_{r_0}^{r} b(G_t)dt}{\int_{r_0}^{r} \tilde{\mu}(G_t)dt},$$

$$\xi^* = \limsup_{r\to\infty} \frac{\int_{r_0}^{r} \chi(G_t)dt}{\int_{r_0}^{r} \tilde{\mu}(G_t)dt}$$

for a fixed r_0. From Theorem 10 we obtain

(11) $$\sum_{\nu=1}^{q} \gamma^*(P_\nu) + b^* \leqq \xi^* - \chi(R).$$

Remark 2. If $\xi = \infty$, (10) is meaningless. On account of Theorem 3 the values of γ, b and ξ do not depend on the choice of μ. Thus to compute them we may choose one particular μ.

The following result was obtained by S. Chern [4]:

Theorem 11. Let S be an open Riemann surface which is obtained from a closed Riemann surface by the deletion of a finite number of points, and let f be a non-constant analytic mapping of S into a closed Riemann surface R. If $\chi(S) \leqq 0$ or $\lim_{n\to\infty} r_n/T_{G_n}(r_n) = 0$ for $\{G_n\}$ and $\{r_n\}$ considered in Theorem 10, then

(12) $$\sum_{\nu=1}^{q} \gamma(P_\nu) + b \leqq -\chi(R)$$

so that R must be a sphere or torus.

Proof. We see that $E_{G_n}(r_n) \leqq \chi(S)r_n$ if n is large. Hence $\xi \leqq 0$ under our assumption, and (12) follows from (10). Theorem 4 implies $\gamma(P_\nu) \geqq 0$ for each ν so that the left hand side of (12) is non-negative. Hence $\chi(R) \leqq 0$ which shows that R is a sphere or a torus.

Remark. If $\chi(S) = 0$ (-1 resp.), then S is conformally equivalent to the surface obtained from a Riemann sphere by the deletion of two (one resp.) points.

We shall study the condition $\lim r_n/T_{G_n}(r_n) = 0$. If this is not true, then there are a finite constant M and a subsequence $\{n_k\}$ of $\{n\}$ such that $T_{G_{n_k}}(r_{n_k}) < Mr_{n_k}$, or $T_{G_{n_k}}(r_{n_k}) = O(r_{n_k})$. We shall prove

Theorem 12. Let S be an open Riemann surface which is obtained from a closed Riemann surface S* by the deletion of a closed set K of logarithmic capacity zero, and let f be a non-constant analytic mapping of S into a closed Riemann surface R. If $T_{G_n}(r_n) = O(r_n)$ for an exhaustion $\{G_n\}$ of S and a sequence of values r_n $(< c_{G_n})$ tending to ∞, then f can be extended to an analytic mapping of S* into R.

Proof. By Theorem 4 we have

$$\int_0^{r_n} n_{G_n}(t, P)dt < T_{G_n}(r_n) + k < Mr_n + k$$

for any $P \in R$ with constants k and M which do not depend on n. We may assume that $k < Mr_1$. It follows that

$$2Mr_n > \int_{r_{n/2}}^{r_n} n_{G_n}(t, P)dt \geqq \frac{r_n}{2} n_{G_n}(\frac{r_n}{2}, P)$$

and hence $n_{G_n}(r_n/2, P) < 4M$. Lemma 9 shows that every subsequence of $\{u_{G_n}\}$ contains a subsequence which converges to a harmonic function locally uniformly on $S - S_0$. Accordingly, given any point $\tilde{P}$ of $S - S_0$ there exists n_0 such that $u_{G_n}(\tilde{P}) < r_n/2$ for every $n \geqq n_0$. Therefore P is covered only finitely often by S. If the cluster set C(f) of f at K contained an open set, then there would be a point of R which is covered infinitely often by S. Hence C(f) is nowhere dense in R.

In case R is an extended plane, by performing a linear transformation, we may assume that f is bounded near K. Taking for granted that every bounded harmonic function defined near a closed set of logarithmic capacity zero can be defined to be harmonic on this set too, we conclude that f can be defined to be an analytic mapping of S* into R. We shall say that K is removable for f.

In case the genus of R is positive, let $w = g$ be a non-constant meromorphic function on R; the existence of such a function is assured by the Riemann-Roch theorem. Then the cluster set of $g \circ f$ at K is nowhere dense in the w-plane. It follows that K is removable for $g \circ f$. Around each point of K there is an open disk $D \subset S^*$ such that the image of $D - K$ by f is contained in a disk on R. We infer that $D \cap K$ is removable for f. Thus K is removable for f. Our theorem is now proved.

A non-constant analytic mapping of S into R is called transcendental if $r_n/T_{G_n}(r_n) \to 0$ for any exhaustion $\{G_n\}$ of S and any sequence of values r_n ($< c_{G_n}$) tending to ∞.

Remark 1. If f is the restriction to S of a non-constant analytic mapping of S* into R, then always $T_{G_n}(r_n) = O(r_n)$.

Remark 2. There exists a parabolic Riemann surface S of infinite genus such that every non-constant analytic mapping of S into a closed Riemann surface is transcendental. See M. Heins [6].

Remark 3. If f cannot be extended to be analytic on S*, then R must be a sphere or a torus. This follows from Théorème I of [8].

Let us observe some consequence of Theorem 11. First let S be $|z| < \infty$ or $0 < |z| < \infty$, and f be a non-constant analytic mapping of S into a closed Riemann surface R. On account of (12) the genus of R is $\leqq 1$. If R is an extended plane or a Riemann sphere (torus resp.), then (12) yields $\sum\gamma(P_\nu) + b \leqq 2$ (= 0 resp.). The (big) Picard theorem follows from this. Let S be an open Riemann surface which is obtained from a closed Riemann surface S* of positive genus by the deletion of a finite number of points. If f is transcendental, then R must be a sphere or a torus. If f is not transcendental, then f is extended to be an analytic mapping of S* into R by Theorem 12.

From Theorem 9 we derive also

Theorem 13. Let S be a parabolic open Riemann surface such that every point of S above P_ν is a branch point with multiplicity $\geqq m_\nu$, and $\{G_n\}$ be any exhaustion. Then there exists $\{r_n\}$ tending to ∞ such that $0 < r_n < c_{G_n}$ for each n and

$$\sum_{\nu=1}^{q} (1 - \frac{1}{m_\nu}) \leqq \xi - \chi(R).$$

Proof. From (9) we obtain

$$\sum_{\nu=1}^{q} (T_G(r) - \overline{N}_G(r, P_\nu)) \leqq E_G(r) - \chi(R)T_G(r) + O(\log T_G(r)),$$

where

$$\overline{N}_G(r, P_\nu) = \int_0^r \overline{n}(t, P_\nu)dt$$

with the number (without counting multiplicity) $\overline{n}(t, P_\nu)$ of points on G_t at which $f = P_\nu$. We have

$$m_\nu \overline{N}_G(r, P_\nu) \leqq N_G(r, P_\nu)$$

so that by Theorem 4

$$\frac{E_G(r)}{T_G(r)} - \chi(R) + O\left(\frac{\log T_G(r)}{T_G(r)}\right) \geqq \sum_{\nu=1}^{q}\left(1 - \frac{1}{m_\nu}\frac{N_G(r, P_\nu)}{T_G(r)}\right)$$

$$\geqq \sum_{\nu=1}^{q}\left(1 - \frac{1}{m_\nu}\frac{T_G(r)+k}{T_G(r)}\right).$$

We derive the required relation easily.

§6. Disk theorem

Let $D_1, \ldots, D_{q'}$ be open disks in R whose closures are mutually disjoint, and denote by R' the domain outside $D_1 \cup \cdots \cup D_{q'}$. Evidently $\chi(R') = \chi(R) + q'$. Set $\ell = \partial D_1 \cup \cdots \cup \partial D_{q'}$. Consider a conformal metric $\rho_z|dz|$ with positive $\rho_z \in C^2$ on R. We assume that $\mu(R) = \iint_R \rho_z^2 dxdy = 1$.

Let D be an arbitrary domain in R whose boundary consists of finitely many analytic closed curves. We set $D' = D \cap R'$. We define $\chi(D')$ as before although D' may not be connected. By Lemma 11 we have

$$\int_{D'} K d\mu = -2\pi\chi(D') - \int_{\partial D \cap R'} d\tau - \int_{\ell \cap D} d\tau - \sum \tau_i,$$

where $\sum \tau_i$ means the sum of the outer angles at the points of intersection of ∂D and ℓ. In particular,

(13) $$\int_{R'} K d\mu = -2\pi(\chi(R) + q') - \int_{\ell} d\tau.$$

Let F be a finite covering surface of R such that the projection of ∂F intersects ℓ only finitely often. We have

$$(14) \qquad \int_{F'} \tilde{K} d\tilde{\mu} = 2\pi\{b(F') - \chi(F')\} - \left(\int_{\partial F'} d\tau + \sum \tau_i\right),$$

where F' is the part of F lying over R'.

We define a Radon measure λ on R starting from a set function defined on the class of open sets $E \subset R$:

$$\lambda(E) = \frac{1}{2\pi}\int_{E\cap R'} K d\mu + \frac{1}{2\pi}\int_{\ell\cap E} d\tau.$$

We note that

$$(15) \qquad \lambda(R) = \frac{1}{2\pi}\int_{R'} K d\mu + \frac{1}{2\pi}\int_{\ell} d\tau = -(\chi(R) + q')$$

by (13). Integrating (3) with respect to λ we have

$$\lambda(R)T(r) + \frac{1}{2\pi}\int_R\int_{\gamma_r-\gamma_0} h(f(\tilde{P}); P, \mu) du^*(\tilde{P}) d\lambda(P) + \int_0^r \tilde{\lambda}(G_t) dt,$$

where $\tilde{\lambda}$ is the pull back of λ to S. By (15)

$$(16) \qquad \begin{aligned} (\chi(R) + q')T(r) = &- \frac{1}{2\pi}\int_0^r\int_{G_t'} \tilde{K} d\tilde{\mu} dt - \frac{1}{2\pi}\int_0^r\int_{\ell_t} d\tau dt \\ &- \frac{1}{2\pi}\int_R\int_{\gamma_r-\gamma_0} h(f(\tilde{P}); P, \mu) du^*(\tilde{P}) d\lambda(P), \end{aligned}$$

where ℓ_t is the part of G_t lying above ℓ. We shall evaluate each integral on the right hand side.

Let Ω be a component of the inverse image $f^{-1}(D_i)$. If it is relatively compact in S, it is called an island. Otherwise, it is called a peninsula. When we exclude a simply connected island, the characteristic increases by one. When we exclude a peninsula or a non-simply connected island, then the characteristic is invariant or decreases. Denote by $m^{(\nu)}(G_t)$ the number of simply connected islands lying above D_ν and included in G_t. Then

$$\chi(G_t') \leqq \chi(G_t) + \sum_{\nu=1}^{q'} m^{(\nu)}(G_t).$$

Denote by $k(G_t)$ the number of intersections of ℓ and the projection of γ_t. Then, for $F = G_t$, the sum $\sum \tau_i$ is not greater than $\pi k(G_t)$.

Next we are concerned with

$$\int_0^r dt \int_{\gamma_t'} d\tau = \int_0^r dt \int_{\gamma_t'} \frac{\partial \log \rho_u}{\partial t} du^*,$$

where γ_t' is the part of γ_t lying above R'. Choose $0 < r_1 < r_2 < \cdots < r_k \leqq r$ so that grad $u \neq 0$ on the part $B_i = \{\tilde{P}; r_i < u(\tilde{P}) < r_{i+1}\}$. On each B_i we can define u^* so that (u, u^*) gives a kind of coordinates. Set $\psi(u, u^*) = \partial \log \rho_u / \partial u$ if $f(\tilde{P}(u, u^*)) \in R'$ and $= 0$ if $f(\tilde{P}(u, u^*)) \notin R'$. We have

$$\int_{r_1}^{r_2} \int_{\gamma_t'} \frac{\partial \log \rho_u}{\partial t} du^* dt = \int_0^1 \int_{r_1}^{r_2} \psi(t, u^*) dt du^*$$

$$= \int_0^1 \sum_i \{\log \rho_u(t_i, u^*) - \log \rho_u(t_{i-1}, u^*)\} du^*,$$

where $U_i(t_{i-1}, t_i)$ coincides with the part of the u^*-level set in R'. The last side is equal to

$$\int_{\gamma_{r_2}'} \log \rho_u du^* - \int_{\gamma_{r_1}'} \log \rho_u du^* + \int_{\ell_{r_1, r_2}} \log \rho_u du^*,$$

where ℓ_{r_1, r_2} is the part of $G_{r_2} - G_{r_1}$ lying above ℓ. By summation we obtain

$$\int_0^r dt \int_{\gamma_{t'}} d\tau = \int_{\gamma_r'} \log \rho_u du^* - \int_{\gamma_0'} \log \rho_u du^* + \int_{\ell_r'} \log \rho_u du^*,$$

where ℓ_r' is the part of $G_r - S_0$ lying above ℓ.

Setting $B'(r) = \int_0^r b(G_t')dt$ and $M^{(\nu)}(r) = \int_0^r m^{(\nu)}(G_t)dt$, we have

$$-\int_0^r \int_{G_t'} \tilde{K} d\tilde{\mu} dt \leqq 2\pi\{E(r) + \sum_{\nu=1}^{q} M^{(\nu)}(r) - B'(r)\} + \pi \int_0^r k(G_t)dt$$

$$+ \int_{\gamma_r'} \log \rho_u du^* - \int_{\gamma_0'} \log \rho_u du^* + \int_{\ell_r'} \log \rho_u du^*$$

by (14). Let us show that the last integral on the right hand side of (16) is bounded. Actually

$$\int_R h(P'; P, \mu) d\lambda(P) = \frac{1}{2\pi}\int_{R'} h(P'; P, \mu) K d\mu(P) + \frac{1}{2\pi}\int_\ell h(P'; P, \mu) d\tau(P).$$

By the aid of Lemma 7 we see easily that $\int_{R'} h(P'; P, \mu) K d\mu(P)$ is a bounded function of P' on R. Secondly

$$\int_\ell h(P'; P, \mu) d\tau(P) = \int_\ell h(P'; P, \mu) \frac{\partial \log \rho_z}{\partial n_P} ds(P).$$

Fix any $P_0 \in \ell$, and let $|z| < 1$ correspond to a disk D_z with center at P_0 such that $D_z \cap \ell$ corresponds to the diameter on the x-axis. Let D_z' correspond to $|z| < 1/2$. By Lemma 7 $h(P'; P, \mu)$ is bounded on $D_z' \times (R - D_z)$ and

$$h(P'; P, \mu) - \log \frac{1}{|z(P') - z(P)|}$$

is bounded on $D_z \times D_z$. We note also that

$$\int_{-1/2}^{1/2} \log \frac{1}{|z(P') - x|} dx \leqq \int_{-1/2}^{1/2} \log \frac{1}{|x|} dx = 1 + \log 2.$$

Since $\partial \log \rho_z/\partial n$ is bounded, we conclude that $\int_\ell h d\tau$ is bounded provided $P' \in D_z'$. Since ℓ is covered by finitely many disks like D_z' and $\int_\ell h d\tau$ is bounded if P' stays away from ℓ, its boundedness on R follows. Thus the last integral in (16) is bounded with respect to r.

Using the fact that Lemma 14 shows the boundedness from below of $\int_{\gamma_0'} \log \rho_u du^*$, we obtain

$$(17)\quad \begin{aligned}(\chi(R) + q')T(r) \leqq E(r) + \sum_{\nu=1}^{q'} M^{(\nu)}(r) - B'(r) + \frac{1}{2}\int_0^r k(G_t)dt \\ + \frac{1}{2\pi}\left(\int_{\gamma_r'} \log \rho_u du^* + \int_{\ell_r'} \log \rho_u du^*\right) + \text{const.}\end{aligned}$$

We set

$$w_1(r) = \int_{\gamma_r'} \log \rho_u du^*,\ w_2(r) = \frac{1}{2}\int_0^r k(G_t)dt,\ w_3(r) = \frac{1}{2\pi}\int_{\ell_r'} \log \rho_u du^*.$$

We shall evaluate them from above.

For r with positive $u^*(\gamma_r') = \int_{\gamma_r'} du^*$ Lemma 15 yields

$$\frac{w_1(r)}{u^*(\gamma_r')} \leqq \frac{1}{2} \log \left(\frac{1}{u^*(\gamma_r')}\int_{\gamma_r'} \rho_u^2 du^*\right)$$

and hence

$$\int_0^t u^*(\gamma_s') \exp\left(\frac{2w_1(s)}{u^*(\gamma_s')}\right)ds \leqq \int_0^t\int_{\gamma_s'} \rho_u^2 du^* ds \leqq \tilde{\mu}(G_t),$$

where we define the integrand on the left hand side to be zero for s with vanishing $u^*(\gamma_s')$. We infer that

$$\int_0^r dt\int_0^t u^*(\gamma_s') \exp\left(\frac{2w_1(s)}{u^*(\gamma_s')}\right)ds \leqq \int_0^r \tilde{\mu}(G_t)dt = T(r).$$

Fix $\beta > 1$. We apply Lemmas 17 and 18 as in the proof of Theorem 9 and have

$$u^*(\gamma_r') \exp\left(\frac{2w_1(r)}{u^*(\gamma_r')}\right) \leqq (T(r))^{\beta^2}$$

outside a certain set with bounded logarithmic length. Accordingly

$$w_1(r) \leqq \frac{1}{2} u^*(\gamma_r')\beta^2 \log T(r) - \frac{1}{2} u^*(\gamma_r') \log u^*(\gamma_r')$$

$$\leqq \frac{\beta^2}{2} |\log T(r)| + \frac{1}{2e} .$$

Let us finally evaluate w_2 and w_3. First we note that

$$|w_2(r)| \leqq \frac{1}{2}\int_{\ell_r} ds_u, \quad |w_3(r)| \leqq \frac{1}{2\pi}\int_{\ell_r} |\log \rho_u| ds_u,$$

where $ds_u = |d(u + iu^*)|$ along ℓ_r. Set

$$f_1(r) = \int_{\ell_r} ds_u, \quad f_2(r) = \int_{\ell_r} \rho_u ds_u, \quad f_3(r) = \int_{\ell_r} \frac{1}{\rho_u} ds_u.$$

Then $f_1^2 \leqq f_2 f_3$ and by Lemma 15

$$2\pi|w_3(r)| \leqq \int_{\ell_r} |\log \rho_u| ds_u \leqq \int_{\ell_r} \log\left(\rho_u + \frac{1}{\rho_u}\right) ds_u$$

$$\leqq f_1(r) \log\left(\frac{1}{f_1(r)}\int_{\ell_r} \left(\rho_u + \frac{1}{\rho_u}\right) ds_u\right)$$

$$\leqq f_1(r) \log (f_2(r) + f_3(r)) + \frac{1}{e} .$$

Thus

$$|w_2(r)| + |w_3(r)| \leqq \sqrt{f_2 f_3}\left(\frac{1}{2} + \frac{1}{2\pi} \log (f_2 + f_3)\right) + O(1).$$

From theorem 4 we infer that

$$(T(r) + C)\int_{\ell} \rho_z|dz| > \int_{\ell} N(r, P)\rho_z|dz|$$

(18)

$$= \int_0^r dt\int_{\ell} n(t, P)\rho_z|dz| = \int_0^r f_2(t)dt.$$

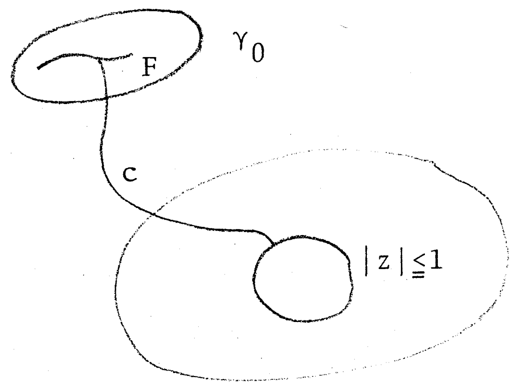
In order to evaluate f_3 we need a technique. We shall apply the following so-called coarea formula; it will be proved in Appendix 8.

Let ϕ be a Lipschitzian function on a bounded domain D in the z-plane. It is known that ϕ is totally differentiable a.e. in D by Rademacher-Stepanov theorem (see [12], p.97, for instance) and grad ϕ is measurable there (see [12], p.87). Let g be a non-negative continuous function on D. Then $\int_{\phi^{-1}(t)} gdm$ is a measurable function of t and

(19) $$\iint_{\phi^{-1}(t)\cap B} gdmdt = \iint_B g|\text{grad } \phi|dxdy$$

for any Borel set $B \subset D$, where m denotes the 1-dimensional Hausdorff measure.

Let $|z| \leqq r$ (> 1) be a local parametric disk such that $|z| < 1$ corresponds to D_ν. Denote by $\delta(z)$ the distance from z in $|z| \leqq r$ to $|z| \leqq 1$, measured with respect to $\rho_z|dz|$. It is a continuous function of z. Set $\delta_0 = \min_{|z|=r} \delta(z) > 0$. Define $G_\nu(t) = \{z;\ |z| \leqq r,\ 0 \leqq \delta(z) < t\}$ and $\ell_\nu(t) = \{z;\ |z| \leqq r,\ \delta(z) = t\}$ for t, $0 < t \leqq \delta_0$. From the definition it follows that $G_\nu(t)$ is a domain. Let $D_\nu(t)$ be the exterior of the unbounded component D_∞ of the complement of $\ell_\nu(t)$. If the exterior is not connected, then there is a curve γ_0 in D_∞ surrounding some subset F of $\ell_\nu(t)$.



We see that $\delta_1 = \min_{z\in\gamma_0} \delta(z) > t$ and $\int_c \rho ds \geqq \delta_1$ for any curve c connecting F and $|z| \leqq 1$. Hence $t = \delta(z) \geqq \delta_1 > t$ on F. This is impossible. Accordingly $D_\nu(t)$ is a simply connected domain. We observe also that $D_\nu(t)$ is the largest domain which contains $|z| \leqq 1$ and whose boundary is contained in $\ell_\nu(t)$. We apply (19) to $B = G_\nu(\delta_0)$, $\phi = \delta$ and $g = 1$, and have

$$\int_0^{\delta_0} m(\ell_\nu(t))dt = \iint_{G_\nu(\delta_0)} |\text{grad } \delta|dxdy.$$

Since δ is Lipschitzian, δ is totally differentiable a.e. and hence $|\text{grad } \delta| = \partial\delta/\partial s \leqq \rho_z$ a.e., where $\partial\delta/\partial s$ is the derivative in the direction of grad δ. It follows that $m(\ell_\nu(t))$ is finite for a.e. t, $0 < t < \delta_0$.

We denote the closure of $G_r - S_0$ by K_r, and take a triangulation of K_r so that the projection is one-to-one on each triangle. We assume that the triangles are mutually disjoint. Accordingly, each triangle may be neither open nor closed. Let $\Delta_1, \ldots, \Delta_k$ be the triangles such that the z-images of their projections $f(\Delta_1), \ldots, f(\Delta_k)$ are not disjoint from $G_\nu(\delta_0)$, and denote by $B_1, \ldots, B_k$ the parts of the images of $f(\Delta_1), \ldots, f(\Delta_k)$ in $G_\nu(\delta_0) - \{|z| \leqq 1\}$. We regard u as a function on $B_1, \ldots, B_k$. We apply again (19) to B_j $(1 \leqq j \leqq k)$, $\phi = \delta$ and $g = |\text{grad } u|^2/\rho_z$ and have

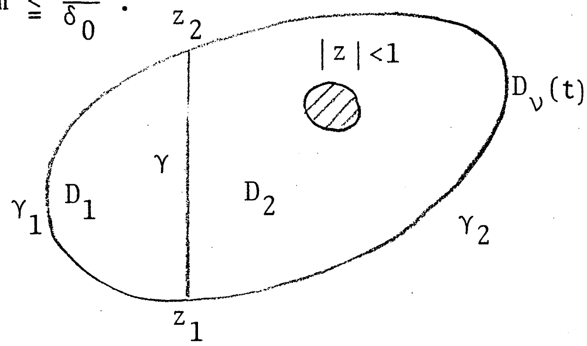
$$\int_0^{\delta_0}\int_{\ell_\nu(t)\cap B_j} \frac{|\text{grad } u|^2}{\rho_z} dmdt \leqq \iint_{B_j} |\text{grad } u|^2 dxdy.$$

Taking a sum we obtain

$$\int_0^{\delta_0}\int_{\ell_\nu(t)\cap K_r} \frac{|\text{grad } u|^2}{\rho_z}\, dmdt \leqq \iint_{K_r} |\text{grad } u|^2 dxdy = r,$$

where $\ell_\nu(t) \cap K_r$ means the part of K_r above $\ell_\nu(t)$. It follows that there exists t, $0 < t < \delta_0$, such that $m(\ell_\nu(t))$ is finite and

$$\int_{\ell_\nu(t)\cap K_r} \frac{|\text{grad } u|^2}{\rho_z}\, dm \leqq \frac{r}{\delta_0} .$$



Let $\gamma = \overline{z_1 z_2}$ be any segment which is a cross-cut of $D_\nu(t)$, and D_1 and D_2 be the domains to which $D_\nu(t)$ is divided by γ. Evidently, $m(\partial D_1) + m(\partial D_2) < \infty$. It is not difficult to see that each $\partial D_i - \gamma$ is, as a closed set, a locally connected continuum. We shall use the arcwise connectedness theorem which asserts that every locally connected continuum is arcwise connected; see p.36 of [13], for instance. It follows that each $\partial D_i - \gamma$ contains a Jordan arc γ_i connecting z_1 and z_2. Each Jordan closed curve $\gamma_i \cup \gamma$ bounds a simply connected domain D_i', and $\gamma_1 \cup \gamma_2$ bounds a domain $D' = D_1' \cup \gamma \cup D_2'$ which evidently includes $D_\nu(t)$. Since $\gamma_1 \cup \gamma_2 \subset \ell_\nu(t)$ and $D_\nu(t)$ is the largest domain which contains $|z| \leqq 1$ and whose boundary is contained in $\ell_\nu(t)$, $D' = D_\nu(t)$ and $\gamma_1 \cup \gamma_2 = \partial D_\nu(t)$. To see that $\gamma_1 \cup \gamma_2$ is simple, take a point $z_0 \in \gamma_2$ outside D_1 and let c_1 be the arc between z_0 and z_2 on ∂D_2. Let z_1' be the point of first intersection of c_1 and γ_1 as in the figure. Assume that $z_1' \neq z_2$, and connect z_1' and z_2 in D_1 by an arc c (dotted line in the

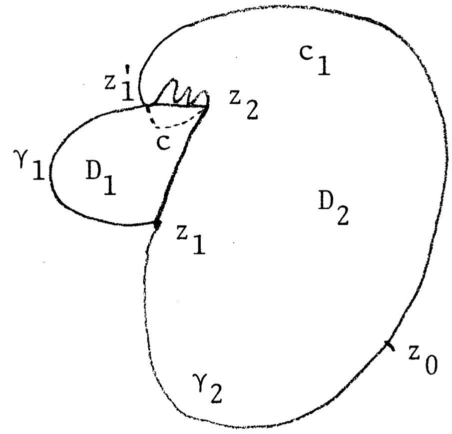


figure). The arc $\widetilde{z_1' z_2} \subset \gamma_2$ and c bound a Jordan domain Δ whose closure contains $\gamma^* = \widetilde{z_1' z_2} \subset \gamma_1$. Since z_1 lies outside of Δ and $\widetilde{z_1 z_0} \subset \gamma_2$ does not meet $\partial\Delta$, $\widetilde{z_1 z_0} \subset \gamma_2$ does not meet γ^*. The curve consisting of $\widetilde{z_1 z_0 z_1'}$, γ^* and $\overline{z_2 z_1}$ is a Jordan closed curve. The union of the domain bounded by this closed curve and D_1 must be equal to $D_\nu(t)$. If we eliminate γ^*, then we obtain a domain which is larger than $D_\nu(t)$ and whose boundary is contained in $\ell_\nu(t)$. This is impossible. Thus c_1 does not meet γ_1 except at z_2. Similarly we see that γ_2 does not meet γ_1 except at z_1 and z_2. Accordingly $\gamma_1 \cup \gamma_2 = \partial D_\nu(t)$ is a rectifiable Jordan closed curve.

Map $|\zeta| < 1$ conformally onto $D_\nu(t)$ by $z = F(\zeta)$. This is extended to a one-to-one continuous function on $|\zeta| \leqq 1$. Denote the image of $|\zeta| = \tau$ $(0 < \tau \leqq 1)$ by $c(\tau)$, and the length of $c(\tau)$ by $L(\tau)$. For an integer $k > 0$ set $\omega = e^{2\pi i/k}$. The function

$$\Lambda_k(\zeta) = |F(\zeta) - F(\omega\zeta)| + \cdots + |F(\omega^{k-1}\zeta) - F(\zeta)|$$

is subharmonic in $|\zeta| < 1$ and continuous on $|\zeta| \leqq 1$ so that it takes its maximum on $|\zeta| = 1$. Therefore

$$\Lambda_k(\zeta) \leqq \max_{|\zeta|=1} \Lambda_k(\zeta) \leqq L(1).$$

As $k \to \infty$ $\Lambda_k(\zeta) \to L(\tau)$ for any ζ on $|\zeta| = \tau$, and hence $L(\tau) \leqq L(1)$ for any τ, $0 < \tau \leqq 1$. Let α be any arc on $|\zeta| = 1$, and set $\alpha' = \{|\zeta| = 1\} - \alpha$. Denote by $\tau\alpha$ and $\tau\alpha'$ the arcs $\{\tau\zeta;\ \zeta \in \alpha\}$ and $\{\tau\zeta;\ \zeta \in \alpha'\}$ respectively. Denote the lengths of their images by $L(F(\tau\alpha))$ and $L(F(\tau\alpha'))$ respectively. Then, for any sequence $\{\tau_n\}$ increasing to 1,

$$L(1) \geqq \liminf_{n\to\infty} L(\tau_n) \geqq \liminf_{n\to\infty} L(F(\tau_n\alpha)) + \liminf_{n\to\infty} L(F(\tau_n\alpha'))$$

$$\geqq L(F(\alpha)) + L(F(\alpha')) = L(1).$$

Accordingly $L(F(\tau_n\alpha)) \to L(F(\alpha))$. We can conclude that $\limsup_{\tau\to 1}\int_{c(\tau)} \psi dm \leqq \int_{\partial D_\nu(t)} \psi dm$ for any non-negative upper semicontinuous function ψ on the closure of $D_\nu(t)$. We denote by $n(P, K_r)$ the number, counted with multiplicity, of points of K_r lying above P. It is an upper semicontinuous function of P. Regarding it as a function on $D_\nu(t)$, we see that there exists τ such that

$$\int_{c(\tau)\cap K_r} \frac{|\text{grad } u|^2}{\rho_z} dm = \int_{c(\tau)} \frac{|\text{grad } u|^2}{\rho_z} n(\cdot, K_r) dm < \frac{2r}{\delta_0}.$$

Denote this $c(\tau)$ by γ'_ν. We see that $|\text{grad } u| dm = ds_u$ along $\gamma'_\nu \cap K_r$ and $\rho_z = \rho_u |\text{grad } u|$. Hence

$$\int_{\gamma'_\nu\cap K_r} \frac{1}{\rho_u} ds_u < \frac{2r}{\delta_0}.$$

Given a domain Ω relatively compact in R' we can find the above $\gamma'_1, \ldots, \gamma'_{q'}$ outside Ω. We denote the interior of γ'_ν by D'_ν, and see that (17) is true for $R - \cup_\nu D'_\nu$. We shall denote the corresponding quantities by $\tilde{E}(r)$, $\tilde{M}(r)$, etc., and by $B(r, \Omega)$ the quantity corresponding to Ω. Then $\tilde{E}(r) = E(r)$, $\tilde{M}(r) \leqq M(r)$, $B(r, \Omega) \leqq \tilde{B}(r) \leqq B(r)$ and

$$\tilde{f}_3(r) = \sum_{\nu=1}^{q'} \int_{\gamma'_\nu\cap K_r} \frac{1}{\rho_u} ds_u < \frac{2q'r}{\delta_0}.$$

We have

$$|\tilde{w}_2(r)| + |\tilde{w}_3(r)| \leqq \text{const.}\sqrt{r\tilde{f}_2(r)}\ \{\log(\tilde{f}_2(r)+r) + \text{const.}\}.$$

By (18) we have

Theorem 14. Let S be a parabolic open Riemann surface which is a covering surface of R, and G and u_G be as before. Let $D_1, \ldots, D_{q'}$ be open disks on R whose closures are mutually disjoint, let $m_G^{(\nu)}(r)$ be the total number of simply connected islands on $G_r = \{P \in G;\ 0 < u_G < r\ (< c_G)\} \cup S_0$ which lie above D_ν, and set $M_G^{(\nu)}(r) = \int_0^r m_G^{(\nu)}(t)dt$. Moreover, let Ω be a domain on R with positive distance from $D_1 \cup \cdots \cup D_{q'}$, denote by $b_G(r, \Omega)$ the sum of the orders of the branch points of G_r above Ω, and set $B_G(r, \Omega) = \int_0^r b_G(t, \Omega)dt$. Define $E_G(r)$ as before. If G contains a fixed $G_0 \supset S_0$, then

$$\sum_{\nu=1}^{q'} (T_G(r) - M_G^{(\nu)}(r)) + B_G(r, \Omega)$$

$$\leqq E_G(r) - \chi(R)T_G(r) + O(\log (T_G(r)+\text{const.})) + O(\sqrt{rw(r)}\log (w(r)+r)),$$

where w(r) satisfies

$$\int_0^r w(t)dt \leqq \text{const.}\ (T_G(r) + \text{const.}).$$

To evaluate w(r) itself from above we give

Lemma 19. Let $\phi(r)$ be a non-negative integrable function defined on $[0, r_0]$. Then there is an interval $I \subset [0, r_0]$ such that $\int_I d \log r < 2$ and

(20) $$r\phi(r) \leqq \max\{re, \psi(r)(\log \psi(r))^2\} \quad \text{on } [0, r_0] - I,$$

where $\psi(r) = \int_0^r \phi(s)ds$.

Proof. Consider $\Psi(t) = t(\log t)^2$ on $[e, \infty)$. Suppose $\psi(r_0) > e$ and define r_1 by $\psi(r_1) = e$. By Lemma 18 there is a set $I_1 \subset [r_1, r_0]$ such that $\int_{I_1} d \log r < 1$ and (20) holds on $[r_1, r_0] - I_1$. On $[1, r_1]$ we have $f(r) \leqq e$ except for a set I_2 with $m(I_2) \leqq 1$. Hence on $[1, r_0]$ we have (20) outside $I_1 \cup I_2$. It is easy to see that $\int_{I_1 \cup I_2} d \log r < 2$. Also in case $\psi(r_0) \leqq e$ we have (20) on $[0, r_0]$ with a similar exception. Our lemma is thus proved.

Noting that $\tilde{\mu}(S_0)r \leqq T(r)$ and applying Lemma 20 to $w(r)$ we obtain

Theorem 15. With the same notation as in Theorem 14 we have

$$\sum_{\nu=1}^{q'} (T_G(r) - M_G^{(\nu)}(r)) + B_G(r, \Omega)$$

$$\leqq E_G(r) - \chi(R)T_G(r) + O(\sqrt{T_G(r)}(\log T_G(r))^2)$$

on $[1, c_G]$ except an interval I_G with $\int_{I_G} d \log r < 2$.

From Theorem 15 we obtain

Theorem 16. (Defect relation) Under the same condition as above, let $\{G_n\}$ be any exhaustion. Then there exists $\{r_n\}$ tending to ∞ such that $0 < r_n < c_{G_n}$ for each n and

$$\sum_{\nu=1}^{q'} \Gamma_\nu + b_\Omega \leqq \xi - \chi(R),$$

where

$$\Gamma_\nu = 1 - \limsup_{n\to\infty} \frac{M_{G_n}^{(\nu)}(r_n)}{T_{G_n}(r_n)}, \quad b_\Omega = \liminf_{n\to\infty} \frac{B_{G_n}(r_n, \Omega)}{T_{G_n}(r_n)},$$

$$\xi = \limsup_{n\to\infty} \frac{E_{G_n}(r_n)}{T_{G_n}(r_n)} .$$

We shall say that S is at least m_ν-ply ramified above D_ν if every simply connected island of S above D_ν has at least $m_\nu (\nu \geqq 1)$ sheets. We establish

Theorem 17. (Disk theorem) Under the same condition as above

$$\sum_\nu \left(1 - \frac{1}{m_\nu^*}\right) \leqq \xi - \chi(R) \tag{21}$$

if S is at least m_ν^*-ply ramified above D_ν, $\nu = 1, \ldots, q'$.

Proof. Denote by $A_{G_t}(D_\nu)$ the mean sheet number of G_t above D_ν. By making use of Ahlfors' covering theorem (see [11; p.140]) we obtain

$$m_\nu^* m_G^{(\nu)}(t) \leqq A_{G_t}(D_\nu) \leqq \mu(G_t) + aL(\gamma_t),$$

where $L(\gamma_t) = \int_{\gamma_t} \rho_u du^*$ and a is a constant depending only on ρ. We see that

$$\int_0^r L(\gamma_t)dt = \int_0^r \int_{\gamma_t} \rho_u du^* dt \leqq \sqrt{r \int_0^r \int_{\gamma_u} \rho_u^2 du^* du} = \sqrt{r\tilde{\mu}(G_r)} .$$

It follows from Lemma 19 that there is a set $I_G \subset [1, c_G]$ such that $\int_{I_G} d \log r < 2$ and $r\tilde{\mu}(G_r) \leqq T_G(r)(\log T_G(r))^2$ on $[1, c_G] - I_G$. Accordingly

$$M_G^{(\nu)}(r) \leqq \frac{1}{m_\nu^*} \left(T_G(r) + a\sqrt{T_G(r)} \log T_G(r)\right)$$

on $[1, c_G] - I_G$. This yields the sum on the left hand side in (21). It is now immediate to conclude our theorem by means of Theorem 16.

Appendix 1. Potentials with kernel h

First we prove

Proposition 1. Let μ be a non-negative measure on R, and fix P_0 on R. If the potential

$$\phi(P, P_0) = \int h(P; P_0, Q)\,d\mu(Q)$$

is of class C^2 in an open set $G \subset R - \{P_0\}$, then $d\mu = (2\pi)^{-1}\Delta\phi\, d\xi d\eta$ in G, where $\Delta\phi = \partial^2\phi/\partial\xi^2 + \partial^2\phi/\partial\eta^2$.

Proof. Let $U \subset G$ be an open disk which corresponds to $|\zeta| < 1$ and for which P_0 is an outer point, and let $V \subset R - U \cup \partial U$ be an open disk with center at P_0. For $P \in U$ we have

$$\phi(P, P_0) = \int_{R-U-V} h(P; P_0, Q)\,d\mu(Q) + \int_V h(P; P_0, Q)\,d\mu(Q)$$

$$+ \int_U \left\{h(P; P_0, Q) + \log \frac{1}{|\zeta(P)-\zeta(Q)|}\right\} d\mu(Q) - \int_U \log \frac{1}{|\zeta(P)-\zeta(Q)|}\,d\mu(Q)$$

By Lemma 3 $h(P; P_0, Q)$ is continuous with respect to $(P, Q) \in U \times (R-U-V)$. Regarding the first integral $\int_{R-U-V} h\,d\mu$ as a function of ζ in U, taking mean values of $\int_{R-U-V} h\,d\mu$ on closed disks in U and applying Fubini's theorem we see that it is a harmonic function of P on U.

To see that the second integral $\int_V h\,d\mu$ is harmonic, suppose $|z| < 1$ corresponds to V and $z = 0$ to P_0. For $Q \in V$ we write

$$h(P; P_0, Q) = h(Q; P_0, P) = \log \frac{1}{|z(Q)|} + u(Q, P),$$

where $u(Q, P)$ is a harmonic function of Q for each $P \in U$. For any $(Q, P) \in V \times U$ we have by the maximum principle

$$\min_{Q\in\partial V} u(Q, P) \leqq u(Q, P) \leqq \max_{Q\in\partial V} u(Q, P).$$

Lemma 3 implies that $u(Q, P) = h(P; P_0, Q)$ is bounded for $(Q, P) \in \partial V \times U$. Accordingly, u is bounded for $(Q, P) \in V \times U$. It follows that $\int_V u d\mu$ is a harmonic function of $P \in U$. Since $\int_V \log |z(Q)| d\mu(Q)$ is constant, $\int_V h d\mu$ is a harmonic function of $P \in U$.

As to the integrand of the third integral we see as in the proof of Lemma 7 that it is bounded on $U \times U$ and hence that the third integral is a harmonic function of $P \in U$. Next, let $g \geqq 0$ be a function of class C^2 such that $g = 1$ on the image D of $|\zeta| < 1/2$ and the support of g is contained in U. We note that ϕ is subharmonic as a function of ζ in $|\zeta| < 1$ so that $\Delta\phi \geqq 0$ there. Set $\rho^2 = g\Delta\phi$ and

$$s(P) = \frac{1}{2\pi}\iint \log \frac{1}{|\zeta - \zeta(P)|} \rho^2(\zeta) d\xi d\eta.$$

It is a well-known classical result that s is of class C^2 as a function of P and its Laplacian is equal to $-\rho^2$; cf., for instance, I. G. Petrovsky: Lectures on partial differential equations, p.219. Thus $\Delta s = -\rho^2 = -g\Delta\phi = -\Delta\phi$ on D, and hence $\Delta(\phi + s) = 0$ on D. Hence $\phi = -s + h'$, where h' is harmonic in D. Thus

$$\int_D \log \frac{1}{|\zeta(P) - \zeta(Q)|} d\mu(Q) = \frac{1}{2\pi}\int_{|\zeta|<1} \log \frac{1}{|\zeta - \zeta(P)|} \rho^2 d\xi d\eta + h'' \text{ in } D,$$

where h'' is harmonic in D. It follows that $d\mu = (2\pi)^{-1}\Delta f d\xi d\eta$ in D; see [3; p.43]. The arbitrariness of U concludes our propoistion.

Proposition 2. There does not exist a measure μ which gives

$$\int h(P; P_0, Q) d\mu(Q) = 1 \qquad \text{on } R.$$

Proof. Suppose this happened. By Proposition 1 $\mu(R - \{P_0\}) = 0$. Hence μ is a point measure $c\varepsilon_{P_0}$ at P_0. If $c > 0$, then $1 = ch(P; P_0, P_0) = \infty$. This is impossible. Accordingly $c = 0$, which is again impossible.

Proposition 2 shows that Ahlfors' requirement that "das Potential der Belegung $S_0(\Omega)$ konstant sein soll" is not possible; see p.5 of [1]. Accordingly some modification of subsequent discussions of Ahlfors is needed.

Appendix 2. Conformal metric

In the beginning of §3 we called $\rho_z|dz|$ a conformal metric if it is subject to $\rho_\zeta|d\zeta| = \rho_z|dz|$ for any change of parameters like $z \to \zeta$. We showed that a positive conformal metric exists on any Riemann surface R.

We shall give special positive conformal metrics. It is known that the universal covering surface R^∞ of R is conformally equivalent to the disk $|w| < 1$ unless R is conformally equivalent to the whole plane $|w| \leqq \infty$ or to $|w| < \infty$ or to $0 < |w| < \infty$ or to a torus. In the case when R^∞ is mapped onto $|w| < 1$ we take

$$\rho(w) = \frac{1}{1 - |w|^2} .$$

If $|w| < 1$ is transformed to $|W| < 1$ by a linear transformation, then

$$\left|\frac{dW}{dw}\right| = \frac{1 - |W|^2}{1 - |w|^2} ;$$

see (1-3) of [2]. Therefore ρ is well-defined on R. In the case when R is conformally equivalent to $|w| \leqq \infty$ or to $|w| < \infty$ or to $0 < |w| < \infty$, we take

$$\rho(w) = \frac{1}{1 + |w|^2} .$$

In the case when R is conformally equivalent to a torus, R^∞ is mapped conformally to $|w| < \infty$. We take $\rho(w) = 1$. We have thus considered $\rho(w) = (1-|w|^2)^{-1}$, $(1+|w|^2)^{-1}$, 1. The corresponding Gaussian curvatures are -4, 4, 0 respectively.

As before we write $d\mu$ for $\rho_z^2 dxdy$. The following theorem is due to S. Chern [4].

Theorem 18. Let P be an arbitrary point on a closed Riemann surface R, and $\rho_z|dz|$ be a conformal metric such that $\rho_z \in C^1$ and $\iint \rho_z^2 dxdy = 1$. Then $s = h(P'; P, \mu)$ is a C^2 solution of the equation $\Delta s = 2\pi\rho_z^2$ on R - {P} such that

$$s(P(z)) + \log |z| \in C^2,$$

where z is a local parameter defined in an open disk U with center at P and z = 0 corresponds to P. Solution is unique up to an additive constant.

Proof. If there are two solutions s_1 and s_2, then $\Delta(s_1 - s_2) = 0$ on R so that $s_1 - s_2$ is constant. We can prove the theorem as Proposition 1 except for the proof of $s + \log |z| \in C^2$ in $|z| < 1$. Let $P' \in U$ and denote its image in $|z| < 1$ by $z(P')$. We have

$$s(P') + \log |z(P')| = \int_{R-U} \{h(P'; P, Q) + \log |z(P')|\} d\mu(Q)$$

$$+ \int_U \left\{ h(P'; P, Q) - \log \left| \frac{z(Q)-z(P')}{z(P')} \right| \right\} d\mu(Q)$$

$$+ \int_U \log |z(Q) - z(P')| d\mu(Q)$$

for $z(P') \neq 0$. The integrand of the second integral is a harmonic function of P' for each $Q \in U$ even at $P' = P$. The required conclusion follows as in the proof of Proposition 1. Our theorem is now proved.

Remark. If there existed a C^2 solution of $\Delta s = 2\pi\rho_z^2$ on the whole R, then s would be subharmonic on R and hence constant. Thus $\Delta s = 0$ which is impossible.

We shall call $h(P'; P, \mu)$ a kernel. As an example we consider Sario's kernel. Let $h = h(P'; P_1, P_2)$ and set

$$s_0 = \log (1 + e^{2h}).$$

This is non-negative and smooth on $R - \{P_1\}$, and has a positive logarithmic pole at P_1. Fix a local parameter z around P_1 so that $z = 0$ corresponds to P_1. We shall denote by $P'(z)$ the mapping of the local parametric disk around $z = 0$. For any other point $P \neq P_1$ we define $h(P'; P, P_1)$ in such a way that

$$h(P'(z); P, P_1) - \log |z| \to \frac{s_0(P)}{2} \qquad \text{as } z \to 0.$$

The function

$$s(P', P) = s_0(P') + 2h(P'; P, P_1)$$

has a positive logarithmic pole at P as its only singularity. This is called Sario's kernel.

Let us see the symmetry $s(P, Q) = s(Q, P)$. Draw small circles C_0, C_0' and C_1 around P, Q and P_1 respectively. By Green's formula

$$\int_{C_0 \cup C_0' \cup C_1} h(\cdot; P, P_1) \frac{\partial h(\cdot; Q, P_1)}{\partial n} ds$$

$$= \int_{C_0 \cup C_0' \cup C_1} h(\cdot; Q, P_1) \frac{\partial h(\cdot; P, P_1)}{\partial n} ds.$$

This gives

$$h(Q; P, P_1) - \frac{s_0(P)}{2} = h(P; Q, P_1) - \frac{s_0(Q)}{2}$$

and hence $s(P, Q) = s(Q, P)$.

We compute

$$\Delta s = \Delta s_0 = \frac{e^{2h}|\text{grad } (2h)|^2}{(1 + e^{2h})^2}$$

in $R - \{P, P_1\}$. We denote it by ρ^2. It is easy to see that ρ is of class C^1 on R. We write ρ as ρ_z when grad is taken with respect to a local parameter z. We observe that $\rho_z|dz|$ is a conformal metric and obtain

$$\iint_R \rho_z^2 dxdy = 4 \int_{-\infty}^{\infty} \int_{c_h} \frac{e^{2h}}{(1 + e^{2h})^2} dh^*dh = 4\pi,$$

where c_h is a level curve for the function h and h^* is a conjugate of h. It is easy to check that the Gaussian curvature of ρ_z is equal to 1. By Chern's theorem (= our Theorem 18) $s(P, Q) = h(P; Q, \mu)$, where $d\mu = \rho_z^2 dxdy$. Sario's kernel has a disadvantage that ρ_z has zero points in general. Actually ρ_z vanishes exactly at the zero points of grad h and the number of the zero points of grad h is equal to the double of the genus of R on account of Lemma 10.

Appendix 3. Zero points of the density ρ_z

Sario's conformal metric in the preceding appendix has zero points at the critical points of grad h. In this and the next appendices we allow isolated zero points of ρ_z. Namely, let $\rho_z > 0$ except at isolated points on R and $\rho_z \in C^2$ outside the zero points. We assume moreover that $\int_R |K| d\mu < \infty$.

Let P_0 be a point on R at which ρ_z vanishes, and z be a local parameter at P_0 such that $|z| < 1$ is a local parametric disk and $z = 0$ corresponds to P_0. Then by Green's formula

$$\lim_{r\to 0} \frac{1}{2\pi}\int_{|z|=r} \frac{\partial \log \rho_z}{\partial r} ds = \frac{1}{2\pi}\int_{|z|<r_0} K d\mu + \frac{1}{2\pi}\int_{|z|=r_0} \frac{\partial \log \rho_z}{\partial r} ds$$

for $0 < r < r_0 < 1$. We call this the order of the zero point P_0 and denote it by $n(\rho_z, P_0)$.

Let us see that $n(\rho_z, P_0)$ is conformally invariant. Suppose $|z| \leqq r_0$ corresponds to a closed region D in a parametric disk $|\zeta| < 1$. We see by the aid of the invariance of $d\tau$ of (4) that

$$n(\rho_z, P_0) = \frac{1}{2\pi}\int_D K d\mu + \frac{1}{2\pi}\int_{\partial D} \frac{\partial \log \rho_\zeta}{\partial n_\zeta} ds_\zeta .$$

Let $|\zeta| \leqq t$, $0 < t < t_0$, be contained in the interior of D. Then we have

$$\frac{1}{2\pi}\int_{|\zeta|=t} \frac{\partial \log \rho_\zeta}{\partial n_\zeta} ds_\zeta$$

$$= \frac{1}{2\pi}\int_{\partial D} \frac{\partial \log \rho_\zeta}{\partial n_\zeta} ds_\zeta + \frac{1}{2\pi}\int_{D-\{|\zeta|<t\}} K d\mu \to n(\rho_z, P_0) \quad \text{as} \quad t \to 0$$

by Green's formula. This shows that $n(\rho_z, P_0)$ does not depend on the choice of a local parameter.

Let $\tilde{P}$ be a branch point of S, and $|\zeta| < 1$ be a local parameter on a neighborhood of $\tilde{P}$ such that $\tilde{P}$ corresponds to $\zeta = 0$. Let $0 < |\zeta_0| < 1$. In some neighborhood of ζ_0, ζ may be regarded as a local parameter on R. Hence $\rho_z^2 dxdy = \rho_\zeta^2 d\xi d\eta$, where $\zeta = \xi + i\eta$, so that $\rho_\zeta = \rho_z|z'(\zeta)|$. As $\zeta \to 0$ $z'(\zeta) \to 0$. Therefore we obtain a conformal metric $\rho^S|d\zeta|$ on S from $\rho_z|dz|$ by defining it to be 0 at the branch points of S and elsewhere in a natural manner. We denote by $\tilde{n}(\rho, S)$ the sum of the orders of the zero points of ρ^S on S. We define $\tilde{n}(\rho, G)$ also for any subdomain G of S.

Let $n(\tilde{P})$ be the multiplicity of f at $\tilde{P}$, and ζ be a local parameter at $\tilde{P}$ such that $\tilde{P}$ corresponds to $\zeta = 0$. Let c_r be the inverse image of $|z| = r$ in the ζ-disk. Then by (4)

$$\begin{aligned}\tilde{n}(\rho_\zeta, \tilde{P}) &\equiv \lim_{r\to 0} \frac{1}{2\pi}\int_{c_r} \frac{\partial \log \rho_\zeta}{\partial n_\zeta} ds_\zeta \\ &= n(\tilde{P})\lim_{r\to 0} \frac{1}{2\pi}\int_{|z|=r} \frac{\partial \log \rho_z}{\partial n_z} ds + \frac{n(\tilde{P})}{2\pi}\int_{|z|=r} d\theta_z - \frac{1}{2\pi}\int_{c_r} d\theta_\zeta \\ &= n(\tilde{P})n(\rho_z, f(\tilde{P})) + n(\tilde{P}) - 1.\end{aligned}$$

Let $\tilde{P}_1, \tilde{P}_2, \ldots$ be the points of G which are projected to the zero points of ρ_z, and set $n(\rho_z, G) = \sum_i n(\tilde{P}_i)n(\rho_z, f(\tilde{P}_i))$. Then

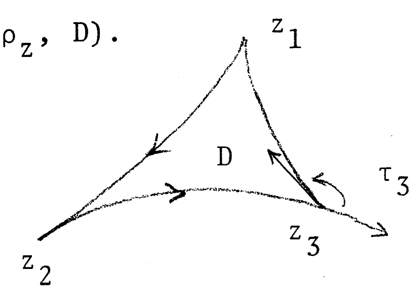
$$\tilde{n}(\rho, G) = \sum_i \tilde{n}(\rho_\zeta, \tilde{P}_i) = n(\rho_z, G) + b(G),$$

where $b(G)$ is the sum of the orders of the branch points of G.

Appendix 4. Gauss-Bonnet's formula

L. Ahlfors [1] beautifully applied Gauss-Bonnet's formula to value distribution theory. We shall show it. Let D be a triangle with corners z_1, z_2, z_3 and with smooth sides in the z-plane such that ρ_z does not vanish on ∂D. We have

$$\iint_D \Delta \log \rho_z dxdy = \int_{\partial D} \frac{\partial}{\partial n} \log \rho_z ds_z - 2\pi n(\rho_z, D).$$



If we denote by θ the angle between the tangent and the x-axis, then

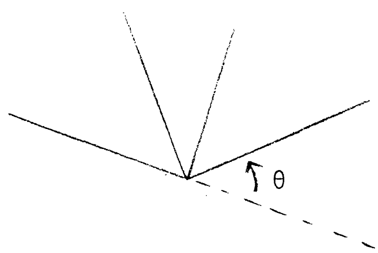
$$\int_{\partial D} d\theta = 2\pi,$$

where the changes of angle at corners are included too. Using $d\tau$ of (4) we obtain

$$\int_D K d\mu = 2\pi - \int_{\partial D} d\tau - \sum \tau_i + 2\pi n(\rho_z, D),$$

where $\int_{\partial D} d\tau$ is taken along ∂D except the corners and τ_1, τ_2, τ_3 are the changes of angle at the corners.

We shall extend this formula to a general subdomain D of R with smooth boundary ∂D on which $\rho_z > 0$. Fix a triangulation so that $\rho_z = 0$ possibly only inside of triangles, and denote the numbers of the corners, sides and triangles by e_0, e_1 and e_2. When j sides issue from a corner in the interior, the sum of the outer angles is equal to $(j - 2)\pi$. Let θ be the variation of the angle at a corner on ∂D. If j sides issue from it, then the sum of the outer angles is equal to $(j - 1)\pi - (\pi - \theta) = (j - 2)\pi + \theta$. Taking a sum we derive



$$\int_D K d\mu = 2\pi\{e_2 - \tfrac{1}{2}\sum (j-2)\} - \int_{\partial D} d\tau - \sum \tau_i + 2\pi n(\rho_z, D),$$

where $\sum \tau_i$ is the sum of the changes of angle at the corners on ∂D. Since $\sum j = 2e_1$, we have

$$(22) \qquad \int_D K d\mu = -2\pi\chi(D) - \int_{\partial D} d\tau - \sum \tau_i + 2\pi n(\rho_z, D),$$

where $\chi(D) = -e_0 + e_1 - e_2$ is the characteristic of D. If D is the whole closed surface R, then there is no ∂D so that

$$(23) \qquad \frac{1}{2\pi}\int_R K d\mu = n(\rho_z, R) - \chi(R).$$

Next we shall establish a formula on a finite covering surface F of R such that the projection of ∂F contains no zero point of ρ_z. Denote by R_k the set of points of R above each of which there are at least k inner points of F; we count the multiplicity for every branch point of F. It follows that ∂R_k contains no zero point of ρ_z. Let $\chi_k = \chi(R_k)$ be the characteristic of R_k. Then $\chi(F) = \sum \chi_k + b(F)$. We obtain

$$(24) \qquad \frac{1}{2\pi}\int_F \tilde{K} d\tilde{\mu} = n(\rho_z, F) + b(F) - \chi(F) - \frac{1}{2\pi}\int_{\partial F} d\tau - \frac{1}{2\pi}\sum \tau_i.$$

This is a generalization of the identity in Lemma 11. We note that we can prove it as in the proof of Lemma 11.

As an application of (22) we give

Second proof of Lemma 10. We may assume $\int_E d\omega^* = 1$. Consider

$$\rho_z = |\text{grad } \omega|$$

and set $A = \{z \in D;\ \text{grad } \omega = 0\}$. Then $\rho_z \in C^2$ and $K = 0$ in $D - A$. Hence $\rho_z|dz|$ is a conformal metric satisfying our conditions.

Taking $\omega + i\omega^*$ as a local parameter on ∂D, we see easily that $\int_{\partial D} d\tau = 0$, so that $\chi(D) = n(\text{grad }\omega, D)$ by (22). This is the required equality.

Appendix 5. Identity for ρ_z with zero points

We shall generalize the identity in Theorem 5 in the case when ρ_z may vanish. First we prove

Lemma 20. Let $g(x, y)$ be a function of class C^2 outside the origin $0 = (0, 0)$ such that g vanishes outside the square $\{|x| < 1/4, |y| < 1/4\}$ and $g(x, y) \to -\infty$ as $(x, y) \to (0, 0)$. If

$$\iint |\Delta g| dxdy < \infty,$$

then g is the sum of the logarithmic potential of density $-(2\pi)^{-1}\Delta g$, a function harmonic at 0 and the potential of a point measure at 0.

Proof. Consider the potential

$$U(x, y) = -\frac{1}{4\pi}\iint \log \frac{1}{(x-\xi)^2+(y-\eta)^2} \Delta g(\xi, \eta) d\xi d\eta.$$

By Poisson's formula $\Delta U = \Delta g$ outside 0 and hence $g = U + h$ outside 0, where h is harmonic outside 0. Denote by M_ϕ^r the mean on the circle $x^2 + y^2 = r^2$ of a function ϕ in general, and set $\phi^+ = \max(\phi, 0)$. For $r \in (0, 1/2)$ we see that

$$U^+(x, y) \leqq |U(x, y)| \leqq \frac{1}{4\pi}\iint \left|\log \frac{1}{(x-\xi)^2+(y-\eta)^2}\right| |\Delta g(\xi, \eta)| d\xi d\eta$$

and

$$M_{U^+}^r \leqq \frac{1}{2\pi} \log \frac{1}{r}\iint |\Delta g| d\xi d\eta.$$

Hence $rM^r_{U^+} \to 0$ as $r \to 0$, and $rM^r_{h^+} \leqq rM^r_{U^+} \to 0$. From a classical result in potential theory (see, e.g., [3; p.196]) it follows that

$$h(x, y) = h_1(x, y) + c \log \frac{1}{r},$$

where h_1 is harmonic at 0 too and $r^2 = x^2 + y^2$. Thus

$$g = U + h_1 + c \log \frac{1}{r}.$$

Let G be a subdomain of S as in §2 and $u = u_G$ be as there. We shall prove

Lemma 21. $\int_{\gamma_t} (\log \rho_u - U)du^*$ is a continuous function of t.

Proof. As in the proof of Lemma 13 we have

$$\log \rho_u = (q-p)\log |w| + \log \rho_z + G(w),$$

where G is a continuous function. By Lemmas 7 and 20 we have

$$\log \rho_z = \frac{1}{2\pi}\int h(f(\tilde{P}); P, \mu)K(P)d\mu(P) + v(z) + c \log \frac{1}{|z|},$$

where v is bounded. Hence

$$\log \rho_u - U = (q-cq-p)\log |w| + \phi(w),$$

where ϕ is bounded. The proof of our lemma is completed as in the proof of Lemma 4.

We prove

Theorem 19.

$$B(r) - E(r) + (\chi(R) - n(\rho_z, R))T(r) + \int_0^r n(\rho_z, G_t)dt$$

$$= \frac{1}{2\pi}\int_{\gamma_r - \gamma_0} (\log \rho_u - U)du^*.$$

Proof. Assume first that the projection of $G_t \cup \gamma_t - G_{t'}$ contains no zero point of ρ_z. By (24) we have

$$\frac{1}{2\pi}\int_{t'}^{t}\int_{G_{t''}} \tilde{K}d\tilde{\mu}dt''$$

$$= \int_{t'}^{t} n(\rho_z, G_{t''})dt'' + \int_{t'}^{t} b(G_{t''})dt'' - \int_{t'}^{t} \chi(G_{t''})dt''$$

$$- \frac{1}{2\pi}\int_{t'}^{t} dt''\int_{\gamma_{t''}} \frac{\partial \log \rho_u}{\partial t''} du^*$$

$$= B(t) - B(t') - E(t) + E(t') + \int_{t'}^{t} n(\rho_z, G_{t''})dt''$$

$$- \frac{1}{2\pi}\int_{\gamma_t - \gamma_{t'}} \log \rho_u du^*.$$

Integrating (3) in Theorem 1 with respect to $Kd\mu$, we derive

$$\{T(t) - T(t')\}\int_R Kd\mu - \int_{\gamma_t - \gamma_{t'}} Udu^* = \int_{t'}^{t} dt''\int_R n(t'', P)K(P)d\mu(P)$$

$$= \int_{t'}^{t}\int_{G_{t''}} \tilde{K}d\tilde{\mu}dt''.$$

We use (23) and obtain

$$B(t) - B(t') - E(t) + E(t') + (\chi(R) - n(\rho_z, R))(T(t) - T(t'))$$

$$+ \int_{t'}^{t} n(\rho_z, G_{t''})dt''$$

$$= \frac{1}{2\pi}\int_{\gamma_t - \gamma_{t'}} (\log \rho_u - U)du^*.$$

There are only finitely many t's such that the projection of γ_t contains some zero points of ρ_z. According to Lemma 21 $\int_{\gamma_t}(\log \rho_u - U)du^*$ is a continuous function of t. We can conclude our theorem easily.

Remark 1. If K is constant, then the right hand side of the identity in the theorem reduces to $(2\pi)^{-1}\int_{\gamma_r - \gamma_0} \log \rho_u du^*$.

Remark 2. We can generalize (7) in Theorem 6 similarly.

Appendix 6. Second proof of the second main theorem

On account of Theorem 3 we may assume that ρ does not vanish on R, that $\rho \in C^2$ and that K = const. on R; such ρ exists as was shown in Appendix 2. Let $g \geqq 0$ be a function on R which is integrable with respect to μ and for which $\int_R g d\mu = 1$. Let C be the constant obtained in Theorem 4; we may and do assume that C is positive. From Theorem 4 we derive

$$T_G(r) + C > \int_R N_G(r, P)g(P)d\mu(P) = \int_0^r dt\int_R n(t, P)g(P)d\mu(P)$$

$$= \int_0^r dt\int_{G_t} \tilde{g}\, d\tilde{\mu} = \int_0^r dt\iint_{G_t} \tilde{g}\, \rho_u^2 du du^* = \int_0^r dt\int_0^t\int_{\gamma_s} \tilde{g}\, \rho_u^2 du^* ds,$$

where u_G is written simply as u and $\tilde{g}$ indicates that g is regarded as a function on S.

We shall apply lemma 15. As λ on S we take $\lambda(E) = \int_{E\cap\gamma_s} du^*$ for Borel set $E \subset S$. By that lemma we have

$$\int_{\gamma_s} \log\ (\tilde{g}\,\rho_u^2)\,du^* \leqq \log \int_{\gamma_s} \tilde{g}\,\rho_u^2 du^*$$

or

$$\int_{\gamma_s} \log \tilde{g}\ du^* + \int_{\gamma_s} \log \rho_u^2 du^* \leqq \log \int_{\gamma_s} \tilde{g}\,\rho_u^2 du^*.$$

It follows that

$$\int_0^r dt \int_0^t \exp\left(\int_{\gamma_s} \log \tilde{g}\ du^* + \int_{\gamma_s} \log \rho_u^2 du^*\right) ds < T_G(r) + C.$$

From Lemma 7 it follows that $h(P'; P, \mu)$ is bounded below on $R \times R$. Choose a constant a' so that $h(P'; P, \mu) + a' > 0$ on R. Set $\sigma(P', P) = h(P'; P, \mu) + a'$ and

$$g = c \exp\left\{2 \sum_{\nu=1}^{q} \sigma(\cdot, P_\nu) + 2 - 2 \log \left(\sum_{\nu=1}^{q} \sigma(\cdot, P_\nu) + 1\right)\right\},$$

where c is chosen so that $\int_R g d\mu = 1$; it is easy to check that g is integrable with respect to μ. Clearly

$$\log \tilde{g} = \log c + 2 \sum_{\nu=1}^{q} \tilde{\sigma}(\cdot, P_\nu) + 2 - 2 \log \left(\sum_{\nu=1}^{q} \tilde{\sigma}(\cdot, P_\nu) + 1\right),$$

where $\tilde{\sigma}(\tilde{P}, P_\nu) = \sigma(f(\tilde{P}), P_\nu)$. Substituting this into the above inequality and applying Lemma 15, we obtain

$$\begin{aligned} T_G(r) + C > \int_0^r dt \int_0^t \exp\Bigg\{\log c &+ 2 \sum_{\nu=1}^{q} \int_{\gamma_s} \tilde{\sigma}(\cdot, P_\nu) du^* + 2 \\ &- 2 \int_{\gamma_s} \log \left(\sum_{\nu=1}^{q} \tilde{\sigma}(\cdot, P_\nu) + 1\right) du^* + \int_{\gamma_s} \log \rho_u^2 du^*\Bigg\} ds \\ \geqq \int_0^r dt \int_0^t \exp\Bigg\{\log c &+ 2 \sum_{\nu=1}^{q} \int_{\gamma_s} \tilde{\sigma}(\cdot, P_\nu) du^* + 2 \\ &- 2 \log \left(\sum_{\nu=1}^{q} \int_{\gamma_s} \tilde{\sigma}(\cdot, P_\nu) du^* + 1\right) + \int_{\gamma_s} \log \rho_u^2 du^*\Bigg\} ds. \end{aligned}$$

For simplicity we denote the last side by $\int_0^r dt\int_0^t e^{w(s)}ds$. We apply Theorem 1 and the equality explained in the Remark to Theorem 5 and derive

$$w(r) \geqq \text{const.} + 4\pi\sum_{\nu=1}^{q}(T_G(r) - N_G(r, P_\nu)) + 2\sum_{\nu=1}^{q}\int_{\gamma_0}\tilde{\sigma}(\cdot, P_\nu)du^*$$

$$- 2\log\left(2\pi\sum_{\nu=1}^{q}(T_G(r) - N_G(r, P_\nu)) + \sum_{\nu=1}^{q}\int_{\gamma_0}\tilde{\sigma}(\cdot, P_\nu)du^* + 1\right)$$

$$+ 4\pi(B_G(r) - E_G(r) + \chi(R)T_G(r))$$

$$\geqq \text{const.} + 4\pi\sum_{\nu=1}^{q}(T_G(r) - N_G(r, P_\nu)) + 2\sum_{\nu=1}^{q}\int_{\gamma_0}\tilde{\sigma}(\cdot, P_\nu)du^*$$

$$- 2\log\left(2\pi qT_G(r) + \sum_{\nu=1}^{q}\int_{\gamma_0}\tilde{\sigma}(\cdot, P_\nu)du^* + 1\right)$$

$$+ 4\pi(B_G(r) - E_G(r) + \chi(R)T_G(r))$$

$$\geqq \text{const.} + 4\pi\sum_{\nu=1}^{q}(T_G(r) - N_G(r, P_\nu)) - 2\log(2\pi qT_G(r) + 1)$$

$$+ 4\pi(B_G(r) - E_G(r) + \chi(R)T_G(r)),$$

where we use the general relation $\alpha - \log(\alpha' + \alpha) \geqq -\log\alpha'$ valid for any $\alpha \geqq 0$ and $\alpha' \geqq 1$. Accordingly,

$$\sum_{\nu=1}^{q}(T_G(r) - N_G(r, P_\nu)) + B_G(r)$$

$$\leqq E_G(r) - \chi(R)T_G(r) + \frac{1}{2\pi}\log T_G(r) + \frac{w(r)}{4\pi} + \text{const.},$$

where $w(r)$ satisfies

$$\int_0^r dt\int_0^t e^{w(s)}ds \leqq T_G(r) + C.$$

Appendix 7. Second main theorem with double integrals

In a private circulation "Remark on a paper of Sario" Wu was concerned with the second main theorem in Sario's form. We shall discuss it here.

We assume that ρ does not vanish on R. Let w be the function defined in the preceding appendix. We know that

$$\int_0^r dt \int_0^t e^{w(s)} ds \leqq T_G(r) + C.$$

Denote by Δ the triangle $\{(s, t);\ 0 \leqq s \leqq r,\ s \leqq t \leqq r\}$. The area is $r^2/2$. We apply Lemma 15 and obtain

$$\begin{aligned}\log (T_G(r) + C) &\geqq \log\left(\iint_\Delta e^{w(s)} dsdt\right) \\ &= \log \frac{r^2}{2} + \log \frac{1}{r^2/2}\iint_\Delta e^{w(s)} dsdt\Big) \\ &\geqq \log \frac{r^2}{2} + \frac{1}{r^2/2}\iint_\Delta w(s)dsdt.\end{aligned}$$

We denote the integrals of T_G, N_G, etc. on Δ by $T_G^{(2)}$, $N_G^{(2)}$, etc. Integrating the inequality last but one in the preceding appendix, we have

$$\sum_{\nu=1}^{q} (T_G^{(2)}(r) - N_G^{(2)}(r, P_\nu)) + B_G^{(2)}(r)$$

$$\leqq E_G^{(2)}(r) - \chi(R)T_G^{(2)}(r) + \frac{1}{2\pi}(\log T_G)^{(2)} + \frac{r^2}{8\pi}(\log T_G(r) + \text{const.}) - \frac{r^2}{8\pi} \log \frac{r^2}{2}$$

$$\leqq E_G^{(2)}(r) - \chi(R)T_G^{(2)}(r) + \frac{r^2}{8\pi} \log T_G(r) + \frac{r^2}{4\pi} \log T_G^{(2)}(r) + O(r^2).$$

Accordingly

$$\sum_{\nu=1}^{q}\left(1 - \frac{N_G^{(2)}(r, P_\nu)}{T_G^{(2)}(r)}\right) + \frac{B_G^{(2)}(r)}{T_G^{(2)}(r)}$$

(25)

$$\leqq \frac{E_G^{(2)}(r)}{T_G^{(2)}(r)} - \chi(R) + \frac{r^2 \log T_G(r)}{8\pi\, T_G^{(2)}(r)} + \frac{r^2 \log T_G^{(2)}(r)}{4\pi\, T_G^{(2)}(r)} + \frac{O(r^2)}{T_G^{(2)}(r)}.$$

To derive a defect relation we observe that

$$T_G^{(2)}(r) \geqq \int_{r/2}^{r}\int_{s}^{r} T_G(t)dtds \geqq T_G(\tfrac{r}{2})\int_{r/2}^{r}\int_{s}^{r} dtds = \frac{r^2}{8} T_G(\tfrac{r}{2}) \geqq \frac{r^3}{16}\tilde{\mu}(S_0)$$

and hence that

$$\frac{r^2 \log T_G^{(2)}(r)}{T_G^{(2)}(r)} = \frac{r^2 \log T_G^{(2)}(r)}{\left(T_G^{(2)}(r)\right)^{2/3}\left(T_G^{(2)}(r)\right)^{1/3}}$$

$$\leqq \left(\frac{16}{\tilde{\mu}(S_0)}\right)^{2/3} \frac{\log T_G^{(2)}(r)}{\left(T_G^{(2)}(r)\right)^{1/4}} \to 0$$

as $0 < r \leqq c_G$ and $r \to \infty$. As in the proof of Theorem 15 or 17 we find a set $I_G \subset [1, c_G]$ such that $\int_{I_G} d \log r < 2$ and

$$r\frac{dT_G^{(2)}(r)}{dt} = r^2 T_G(r) \leqq T_G^{(2)}(r)\left(\log T_G^{(2)}(r)\right)^2$$

on $[1, c_G] - I_G$. Therefore

$$\frac{r^2 \log T_G(r)}{T_G^{(2)}(r)} \leqq \frac{r^2}{T_G^{(2)}(r)}\left(\log T_G^{(2)}(r) + 2 \log\log T_G^{(2)}(r)\right) \to 0$$

as $r \in [1, c_G] - I_G$ and $r \to \infty$.

Let $\{G_n\}$ be any exhaustion, and choose $\{r_n\}$ such that $c_{G_n}/2 < r_n < c_{G_n}$ and $r_n \in [1, c_{G_n}] - I_{G_n}$ for each n. Set

$$\gamma^{(2)}(P_\nu) = 1 - \limsup_{n\to\infty} \frac{N_{G_n}^{(2)}(r_n, P_\mu)}{T_{G_n}^{(2)}(r_n)}, \quad b^{(2)} = \limsup_{n\to\infty} \frac{B_{G_n}^{(2)}(r_n)}{T_{G_n}^{(2)}(r_n)},$$

$$\xi^{(2)} = \limsup_{n\to\infty} \frac{E_{G_n}^{(2)}(r_n)}{T_{G_n}^{(2)}(r_n)}.$$

From (25) there follows the relation

$$\sum_{\nu=1}^{q} \gamma^{(2)}(P_\nu) + b^{(2)} \leqq \xi^{(2)} - \chi(R).$$

We note that each $\gamma^{(2)}(P_\nu) \geqq 0$ because $N_{G_n}^{(2)}(r_n, P_\nu) \leqq T_{G_n}^{(2)}(r_n) + Cr_n^2/2$ by Theorem 4.

If we use the existence of p as in Remark 1 in §5 and define

$$\gamma^{*(2)}(P_\nu) = 1 - \limsup_{r\to\infty} \frac{\int_0^r\int_s^r\int_{t_0}^t n(\tau, P_\nu)d\tau dtds}{\int_0^r\int_s^r\int_{t_0}^t \tilde{\mu}(G_\tau)d\tau dtds}, \qquad \text{etc.},$$

then

$$\sum_{\nu=1}^{q} \gamma^{*(2)}(P_\nu) + b^{*(2)} \leqq \xi^{*(2)} - \chi(R).$$

From this we can derive (11) by using a generalized form of l'Hôpital's rule (cf. [14; p.518]).

Remark. Chern [4] and Wu [14] used $g = c_\alpha\{\sum_{\nu=1}^q \exp s(\cdot, P_\nu)\}^{2\alpha}$, $0 < \alpha < 1$, where c_α is determined so that $\int_R g\, d\mu = 1$. It seems, however, that we meet a diffculty because $c_\alpha \downarrow 0$ as $\alpha \uparrow 1$.

Appendix 8. Proof of coarea formula

We shall prove the formula (19). All functions will be real-valued in this section. First we recall the definition of Hausdorff measure. Given a set X on a line or in a plane and $\varepsilon > 0$, set

$$m^{(\varepsilon)}(X) = \inf_{\Delta_\varepsilon} \sum_i \operatorname{diam} X_i \quad \text{and} \quad m_2^{(\varepsilon)}(X) = \frac{\pi}{4} \inf_{\Delta_\varepsilon} \sum_i (\operatorname{diam} X_i)^2,$$

where Δ_ε is a division of X into mutually disjoint sets X_1, X_2, ... of diameter less than ε. The limits $m(X)$ and $m_2(X)$ of $m^{(\varepsilon)}(X)$ and $m_2^{(\varepsilon)}(X)$ as $\varepsilon \to 0$ are the one and two-dimensional Hausdorff measures respectively. It is easy to see that m is equal to the Lebesgue linear outer measure on a line. For a set X on a plane $m(X) = 0$ if and only if X is of Lebesgue measure zero.

In general, an upper integral $\overline{\int} f(t)dt$ is defined for any $f \geqq 0$ by $\inf \int g(t)dt$ for measurable $g \geqq f$. It is easy to find a measurable function $f' \geqq f$ with $\int f'dt = \overline{\int} fdt$. It follows that $\overline{\int} f_n dt \uparrow \overline{\int} fdt$ if $f_n \uparrow f$.

We begin with

Lemma 22. Let ϕ be a Lipschitzian function with Lipschitz constant c defined on a bounded Borel set B in the (x, y)-plane. Then

$$\overline{\int} m(\phi^{-1}(t))dt \leqq \frac{4c}{\pi} m_2(B).$$

Proof. Given $\varepsilon > 0$, choose a countable division Δ_ε of B such that $\operatorname{diam} A < \varepsilon$ for every $A \in \Delta_\varepsilon$ and

$$\sum_{A \in \Delta_\varepsilon} \frac{\pi}{4} (\operatorname{diam} A)^2 \leqq m_2^{(\varepsilon)}(B) + \varepsilon.$$

Set $\Delta_{\varepsilon,t} = \{A \in \Delta_\varepsilon;\ t \in \phi(A)\}$. Then $\phi^{-1}(t) \subset \cup\{A;\ A \in \Delta_{\varepsilon,t}\}$. We have

$$m^{(\varepsilon)}(\phi^{-1}(t)) \leqq \sum_{A\in\Delta_{\varepsilon,t}} \text{diam } A = \sum_{A\in\Delta_\varepsilon} (\text{diam } A)\chi_{\phi(A)}(t),$$

where $\chi_{\phi(A)}$ denotes the characteristic function of $\phi(A)$. It follows that

$$\overline{\int} m^{(\varepsilon)}(\phi^{-1}(t))dt \leqq \sum_{A\in\Delta_\varepsilon} \text{diam } A \overline{\int}\chi_{\phi(A)}(t)dt$$

$$\leqq \sum_{A\in\Delta_\varepsilon} (\text{diam } A)\text{diam } \phi(A) \leqq c \sum_{A\in\Delta_\varepsilon} (\text{diam } A)^2$$

$$\leqq \frac{4c}{\pi} (m_2^{(\varepsilon)}(B) + \varepsilon) \leqq \frac{4c}{\pi} (m_2(B) + \varepsilon).$$

As $\varepsilon \downarrow 0$ $m^{(\varepsilon)}(\phi^{-1}(t)) \uparrow m(\phi^{-1}(t))$. Therefore

$$\overline{\int} m(\phi^{-1}(t))dt \leqq \frac{4c}{\pi} m_2(B).$$

This completes the proof.

Let $\phi(x, y)$ be a function defined on a Borel set B in the (x, y)-plane. We call it totally differentiable at a non-isolated point (x_0, y_0) relative to B if we can write

$$\phi(x, y) = \phi(x_0, y_0) + a(x-x_0) + b(y-y_0) + o(\sqrt{(x-x_0)^2+(y-y_0)^2}).$$

We shall write $\phi_x^{(B)}$ and $\phi_y^{(B)}$ or simply ϕ_x and ϕ_y for a and b respectively. If (x_0, y_0) is isolated, then we set $\phi_x = \phi_y = 0$ at (x_0, y_0). We shall say that ϕ is totally differentiable (everywhere) on a set B if ϕ is so at every non-isolated point of B.

Next we prove

Lemma 23. Let $\phi(x, y)$ be a Lipschitzian function which is defined on a bounded Borel set B_0 in the (x, y)-plane and totally differentiable relative to B_0 a.e. on B_0. Assume that $m(\phi^{-1}(t) \cap K)$ is a measurable function of t and

$$\int m(\phi^{-1}(t) \cap K)dt = \iint_K |\text{grad } \phi| dxdy$$

for every compact set $K \subset B_0$ such that ϕ is totally differentiable relative to B_0 everywhere on K, ϕ_x and ϕ_y are continuous as functions on K and

$$(26) \qquad \lim_{r\to 0} \sup_{\substack{0<|z'-z|<r \\ z,z'\in K}} \frac{|\phi(z')-\phi(z)-\text{grad } \phi\cdot(z'-z)|}{|z'-z|} = 0,$$

where $z = (x, y)$, $z' = (x', y')$ and $z' - z$ is regarded as a vector. Then $m(\phi^{-1}(t) \cap B)$ is a measurable function of t and

$$(27) \qquad \int m(\phi^{-1}(t) \cap B)dt = \iint_B |\text{grad } \phi| dxdy$$

for any Borel set $B \subset B_0$.

Proof. Let a Borel set $B \subset B_0$ be given. Using Lusin's and Egorov's theorems we can find a compact set $K_1 \subset B$ such that $m_2(B-K_1) < 1/2$, ϕ is totally differentiable on K_1 relative to B_0, ϕ_x and ϕ_y are continuous as functions on K_1 and (26) is true for $K = K_1$. Similarly we can find a subset K_2 of $B - K_1$ such that ϕ is totally differentiable on K_2 relative to B_0, ϕ_x and ϕ_y are continuous on K_2, (26) is true for $K = K_2$ and $m_2(B-K_1-K_2) < 1/2^2$. We continue this process and set $B' = B - K_1 - K_2 - \cdots$. Evidently $m_2(B') = 0$. By our assumption $m(\phi^{-1}(t) \cap K_n)$ is a measurable function of t for each n and (27) is true for K_1, K_2, ... From Lemma 23 it follows that $m(\phi^{-1}(t) \cap B')$ is a measurable function of t (actually $m(\phi^{-1}(t) \cap B') = 0$ for a.e. t) and (27) is true. Thus $m(\phi^{-1}(t) \cap B)$ is a measurable function of t and (27) is true for B.

Lemma 24. Let ϕ be a continuous function defined on a Borel set B_0 in the (x, y)-plane, and g be a non-negative Borel measurable function on B_0. If $m(\phi^{-1}(t) \cap B)$ is a Borel measurable function of t for every Borel set $B \subset B_0$, then $\int_{\phi^{-1}(t)} gdm$ is also a Borel measurable function of t. Moreover, let h be a non-negative measurable function on B_0. If

$$\int m(\phi^{-1}(t) \cap B)dt = \iint_B h(x, y)dxdy$$

for every Borel set $B \subset B_0$, then

$$\iint_{\phi^{-1}(t)} gdmdt = \iint_{B_0} g(x, y)h(x, y)dxdy.$$

Proof. Set

$$E_q^{(p)} = \{(x, y) \in B_0;\ \frac{q-1}{2^p} \leqq g(x, y) < \frac{q}{2^p}\} \qquad (q = 1, \ldots, 2^{2p})$$

and

$$E^{(p)} = \{(x, y) \in B_0;\ 2^p \leqq g(x, y)\}.$$

Define g_p on B_0 by $(q-1)/2^p$ on $E_q^{(p)}$ and 2^p on $E^{(p)}$. Then $\int_{\phi^{-1}(t)} g_p dm$ is a Borel measurable function of t. As $p \to \infty$ $g_p \uparrow g$ and the measurability of $\int_{\phi^{-1}(t)} gdm$ is concluded. Moreover,

$$\iint_{\phi^{-1}(t)} g_p dmdt = \sum_{q=1}^{2^{2p}} \frac{q-1}{2^p}\int m(\phi^{-1}(t) \cap E_q^{(p)})dt$$

$$+ 2^p\int m(\phi^{-1}(t) \cap E^{(p)})dt = \sum_{q=1}^{2^{2p}} \frac{q-1}{2^p}\iint_{E_q^{(p)}} hdxdy + 2^p\iint_{E^{(p)}} hdxdy$$

$$= \iint_{B_0} g_p hdxdy.$$

By letting $p \to \infty$ we obtain the required equality.

Lemma 25. Let $\phi = (u, v)$ be a Lipschitzian transformation of a compact set K in the (x, y)-plane into the (u, v)-plane such that u and v are totally differentiable on K relative to K, u_x, u_y, v_x, v_y are continuous on K, the Jacobian $J\phi$ does not vanish on K and

$$\lim_{r\to 0} \sup_{\substack{0<|z'-z|<r \\ z,z'\in K}} \frac{|\phi(z')-\phi(z)-D\phi(z)(z'-z)|}{|z'-z|} = 0, \tag{28}$$

where $D\phi$ is the Jacobian matrix of ϕ and $D\phi(z)(z'-z)$ is a vector in the (u, v)-plane. Then there exists $r_0 > 0$ such that, for every point $z = (x, y)$ of K, ϕ is one-to-one on $K \cap \Delta(z, r_0)$ and ϕ^{-1} is a Lipschitzian transformation of $\phi(K \cap \Delta(z, r_0))$, where $\Delta(z, r_0) = \{z';\ |z'-z| < r_0\}$. Moreover, ϕ^{-1} is totally differentiable on $\phi(K \cap \Delta(z, r_0))$ and

$$D\phi^{-1} = \frac{1}{J\phi}\begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix}. \tag{29}$$

Proof. Set $a = \min\left(|J\phi|/\sqrt{u_x^2 + u_y^2 + v_x^2 + v_y^2}\right)$ on K. Take any $z_0 \in K$, and let A be the inverse matrix of $D\phi(z_0)$. Given different z and z', set $\zeta = D\phi(z_0)z$ and $\zeta' = D\phi(z_0)z'$. We have $|A(\zeta'-\zeta)| \leqq |\zeta'-\zeta|/a$ and hence

$$\frac{|\zeta'-\zeta|}{|z'-z|} = \frac{|\zeta'-\zeta|}{|A(\zeta'-\zeta)|} \geqq \frac{a|\zeta'-\zeta|}{|\zeta'-\zeta|} = a > 0.$$

Assume that $z, z' \in K$, $|z - z_0| < r$ and $|z' - z_0| < r$. We write

$$\phi(z') - \phi(z) = D\phi(z)(z'-z) + o(|z'-z|)$$

$$= D\phi(z_0)(z'-z) + (D\phi(z) - D\phi(z_0))(z'-z) + o(|z'-z|),$$

where $o(|z'-z|)$ is a vector. On account of (28) there exists $\varepsilon_1(r)$ which tends to 0 as $r \to 0$ and which satisfies

$$|o(z'-z)| \leqq |z' - z|\varepsilon_1(r).$$

Since $D\phi$ is uniformly continuous on K, there exists $\varepsilon_2(r)$ which tends to 0 as $r \to 0$ and which dominates the norm of $D\phi(z) - D\phi(z_0)$. By setting $\varepsilon(r) = \varepsilon_1(r) + \varepsilon_2(r)$ we have

$$|\phi(z') - \phi(z) - (\zeta'-\zeta)|$$

(30)

$$= |\phi(z') - \phi(z) - D\phi(z_0)(z'-z)| \leqq |z' - z|\varepsilon(r).$$

It follows that

$$\frac{|\phi(z')-\phi(z)|}{|z'-z|} \geqq \left|\frac{\zeta'-\zeta}{z'-z}\right| - \varepsilon(r) \geqq a - \varepsilon(r) > \frac{a}{2}$$

if r is small. Thus there is r_0 such that ϕ is one-to-one on $K \cap \Delta(z_0, r_0)$, and the corresponding inverse transformation is Lipschitzian.

To prove the latter part of the lemma, take z, z' on $K \cap \Delta(z_0, r_0)$, set $w = \phi(z)$ and $w' = \phi(z')$, and write D^{-1} for $(D\phi(z))^{-1}$. We note that

$$D^{-1} = \frac{1}{J\phi}\begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix}.$$

We have

$$w' - w = D\phi(z)(z'-z) + o(|z'-z|)$$

and hence

$$D^{-1}(w'-w) = z' - z + D^{-1}o(|z'-z|) = \phi^{-1}(w') - \phi^{-1}(w) + D^{-1}o(|z'-z|).$$

Since

$$|D^{-1}o(|z'-z|)| \leqq \frac{1}{a}|o(|z'-z|)|$$

and $|z' - z|/|w' - w| < 2/a$, we obtain $D^{-1}o(|z'-z|) = o(|w'-w|)$.
Hence

$$\phi^{-1}(w') = \phi^{-1}(w) + D^{-1}(w'-w) + o(|w'-w|).$$

This shows that ϕ^{-1} is totally differentiable at every point of $\phi(K \cap \Delta(z_0, r_0))$ and $D\phi^{-1} = D^{-1}$. Thus $D\phi^{-1}$ has the required form.

Lemma 26. Let $\phi(t) = (u(t), v(t))$ be a Lipschitzian transformation of a compact set K on a line into the (u, v)-plane such that $u'(t)$ and $v'(t)$ relative to K exist and are continuous, $(u'(t), v'(t)) \neq 0$ on K and

$$\lim_{r\to 0} \sup_{\substack{0<|t'-t|<r \\ t,t'\in K}} \frac{|\phi(t')-\phi(t)-(t'-t)\phi'(t)|}{|t'-t|} = 0,$$

where $\phi'(t) = (u'(t), v'(t))$. Then there exist $d_0 > 0$ and a function ε_d of d in $(0, d_0)$ such that $\varepsilon_d \to 0$ as $d \to 0$ and

$$(31) \qquad (1 - \varepsilon_{d_B})|\phi'(t)|m(B) \leqq m(\phi(B)) \leqq (1 + \varepsilon_{d_B})|\phi'(t)|m(B)$$

for every Borel set $B \subset K$ with diameter $d_B < d_0$, where t is an arbitrary point of B.

Proof. We observe that a relation similar to (30) is true for ϕ in the present lemma. Namely,

$$(32) \qquad |\phi(t') - \phi(t) - (t'-t)\phi'(t_0)| \leqq |t' - t|\varepsilon(r)$$

at any point $t_0 \in B$, where $r = \max(|t - t_0|, |t' - t_0|)$ and $\varepsilon(r) \downarrow 0$ as $r \downarrow 0$. It follows as in Lemma 25 that there exists $d_0 > 0$ such that, for every $t \in K$, ϕ is one-to-one on $K \cap (t-d_0, t+d_0)$ and ϕ^{-1} is Lipschitzian on $\phi(K \cap (t-d_0, t+d_0))$. We can choose a common Lipschitz constant $b > 0$ for all $t \in K$. Take any non-empty Borel set $B \subset K$ of diameter less than d_0, and fix an arbitrary point $t_0 \in B$. Take t, t' arbitrarily on $K \cap (t_0-d_0, t_0+d_0)$. Then by (32) we have

$$\frac{|t' - t|}{|\phi(t')-\phi(t)|} \leqq \frac{|\phi(t') - \phi(t)| + |t' - t|\varepsilon(r)}{|\phi'(t_0)||\phi(t') - \phi(t)|} \leqq \frac{1 + b\varepsilon(r)}{|\phi'(t_0)|} .$$

It follows that

$$m(\phi(B)) \geqq (1+b\varepsilon(d_B))^{-1}|\phi'(t_0)|m(B) \geqq (1-b\varepsilon(d_B))|\phi'(t_0)|m(B).$$

We have also by (32)

$$\frac{|\phi(t') - \phi(t)|}{|t' - t|} \leqq |\phi'(t_0)|\left(1 + \frac{\varepsilon(r)}{|\phi'(t_0)|}\right)$$

and

$$m(\phi(B)) \leqq \left(1 + \frac{\varepsilon(d_B)}{|\phi'(t_0)|}\right)|\phi'(t_0)|m(B).$$

Thus we obtain (31) with

$$\varepsilon_{d_B} = \max\left\{b\varepsilon(d_B), \frac{\varepsilon(d_B)}{\min\limits_{t\in K}|\phi'(t)|}\right\} ,$$

where d_B is restricted to be less than d_0.

Lemma 27. Let $\phi(t)$ be as in Lemma 26. If it is one-to-one, then

$$m(\phi(B)) = \int_B |\phi'(t)|dt$$

for any Borel subset B of K.

Proof. Let d_0 be the constant in Lemma 26 and B' be a non-empty Borel subset of K. Fix an arbitrary point t_0 in B'. Since $\phi'(t)$ is continuous on K, by (31) we have

$$m(\phi(B')) = \int_{B'} |\phi'(t)|dt + \int_{B'} (|\phi'(t_0)| - |\phi'(t)|)dt$$

$$+ \eta|\phi'(t_0)|m(B') = \int_{B'} |\phi'(t)|dt + \eta' m(B'),$$

where η and η' tend to zero as diam $B' \to 0$. Dividing B into finitely many Borel sets $\{B_i\}$ of diameter less than $\delta < d_0$, we obtain

$$m(\phi(B)) = \sum_i m(\phi(B_i)) = \sum_i \int_{B_i} |\phi'(t)|dt + \eta'' m(B)$$

$$= \int_B |\phi'(t)|dt + \eta'' m(B),$$

where $\eta'' \to 0$ as $\delta \to 0$. By letting $\delta \to 0$ we derive $m(\phi(B)) = \int_B |\phi'(t)|dt$.

As the last lemma in this section we shall prove a formula for change of variables.

Lemma 28. Let $\psi = (u, v)$ be a one-to-one Lipschitzian transformation of a Borel set B in the (x, y)-plane into the (u, v)-plane such that u and v are totally differentiable on B relative to B, u_x, u_y, v_x, v_y are continuous on B, (28) is true and the Jacobian J of ψ does not vanish on B. Then for any continuous function $\phi \geqq 0$ on $\psi(B)$,

$$\iint_{\psi(B)} \phi \, du dv = \iint_B \phi(\psi(z))|J(z)| dxdy. \tag{33}$$

Proof. We may assume that B is a compact set K. Given $\varepsilon > 0$, cover K by small open squares $\{S_n\}$ so that $K \cap S_n \neq \emptyset$ for each n and $\sum |S_n| - |K| < \varepsilon$, where $|\ |$ means a Lebesgue measure. Take an arbitrary point $z_n \in K \cap S_n$. By the transformation $\psi(z_n) + D\psi(z_n)(z-z_n)$, S_n is mapped to a parallelogram with area $|J(z_n)||S_n|$. We may assume that the ψ-image of each $K \cap S_n$ is contained in a parallelogram with area $(|J(z_n)|+\varepsilon)|S_n|$, and that $\sup_{\psi(K\cap S_n)} \phi - \inf_{\psi(K\cap S_n)} \phi < \varepsilon$. Hence $|\psi(K \cap S_n)| \leqq (|J(z_n)|+\varepsilon)|S_n|$. Set $B_1 = K \cap S_1$, $B_2 = K \cap (S_2 - S_1)$, $B_3 = K \cap (S_3 - S_1 - S_2)$, ..., $M = \max_{z\in K}|J(z)|$ and $M' = \max_{w\in\psi(K)} \phi(w)$. We have

$$\iint_{\psi(K)} \phi \, du dv \leqq \sum_n \sup_{\psi(K\in S_n)} \phi(|J(z_n)|+\varepsilon)|S_n|$$

$$\leqq \sum_n \sup_{\psi(K\cap S_n)} \phi |J(z_n)||B_n| + M M' \sum_n (|S_n|-|B_n|) + \varepsilon M' \sum_n |S_n|$$

$$\leqq \sum_n \sup_{K\cap S_n} \phi\circ\psi |J(z_n)||B_n| + MM'\varepsilon + M'(|B|+\varepsilon)\varepsilon.$$

Since $\sum_n \sup_{K\cap S_n} \phi\circ\psi|J(z_n)||B_n| \to \iint_K \phi\circ\psi|J(z)|dxdy$ as $\max_n$ diam $S_n \to 0$, we obtain

$$\iint_{\psi(K)} \phi \, du dv \leqq \iint_K \phi\circ\psi|J(z)|dxdy.$$

By Lemma 25 ψ^{-1} has the same properties as ψ, and $1/J(z)$ is the Jacobian of ψ^{-1}. Considering $\phi^* = \phi\circ\psi|J|$ on $\psi(K)$ we have

$$\iint_K \phi\circ\psi|J|dxdy = \iint_{\psi^{-1}\circ\psi(K)} \phi^* dxdy$$

$$\leqq \iint_{\psi(K)} \phi^*\circ\psi^{-1}|J|^{-1}dudv = \iint_{\psi(K)} \phi dudv.$$

This is the inverse inequality. Thus (33) is derived.

Proof of (19). In view of Lemmas 23 and 24 it suffices to establish

$$\int m(\phi^{-1}(t)\cap K)dt = \iint_K |\text{grad } \phi|dxdy \tag{34}$$

for any compact set $K \subset D$ with the property that ϕ is totally differentiable everywhere on K, ϕ_x and ϕ_y are continuous on K and

$$\lim_{r\to 0} \sup_{\substack{0<|z'-z|<r \\ z,z'\in K}} \frac{|\phi(z')-\phi(z)-\text{grad } \phi(z'-z)|}{|z'-z|} = 0,$$

where $z' - z$ is regarded as a vector. Set

$$K_0 = \{z \in K;\ \phi_x = \phi_y = 0\},$$

and let z_0 be any point of $K - K_0$. Suppose $\phi_x \neq 0$ at z_0, and let Δ_0 be a closed disk around z_0 such that $\phi_x \neq 0$ on $K\cap\Delta_0$. Define a mapping ψ of $K\cap\Delta_0$ into the ζ-plane by (ϕ, y). Then it is Liptschitzian and its Jacobian at $z \in K\cap\Delta_0$ is equal to $\phi_x(z)$. Moreover, it has the same properties as ϕ on K. Namely, each component of $\psi = (\phi, y)$ is totally differentiable, ψ_x and ψ_y are continuous and

$$\lim_{r\to 0} \sup_{\substack{0<|z'-z|<r \\ z,z'\in K}} \frac{|\psi(z')-\psi(z)-D\psi(z)(z'-z)|}{|z'-z|} = 0.$$

By means of Lemma 25 choose an open disk $\Delta \subset \Delta_0$ with center at z_0 such that ψ is one-to-one on $K \cap \Delta$ and the inverse transformation ψ^{-1} is totally differentiable on $\psi(K \cap \Delta)$. Set

$$E_\xi = \{\eta;\ (\xi, \eta) \in \psi(K \cap \Delta)\}$$

and define a mapping θ_ξ of E_ξ into a plane by

$$\theta_\xi(\eta) = \psi^{-1}(\xi, \eta) \in K \cap \Delta.$$

It is one-to-one, and $d\theta_\xi/d\eta = (-\phi_y, \phi_x)/\phi_x$ in view of (29). We note that $d\theta_\xi/d\eta$ is continuous and does not vanish.

By Lemma 27 we have

$$m(\phi^{-1}(\xi) \cap B') = \int_{E_\xi \cap \psi(B')} \left|\frac{d\theta_\xi(\eta)}{d\eta}\right| d\eta$$

for any Borel set $B' \subset K \cap \Delta$. Since $|d\theta_\xi(\eta)/d\eta| = |\text{grad } \phi|/|\phi_x|$ and this may be regarded as a continuous function on $\psi(B')$,

$$\iint_{\psi(B')} \left|\frac{d\theta_\xi(\eta)}{d\eta}\right| d\eta d\xi$$

exists and is finite. By Fubini's theorem $m(\phi^{-1}(t) \cap B')$ is a measurable function of t, and by Lemma 28

$$\int m(\phi^{-1}(t) \cap B')dt = \iint_{\psi(B')} \frac{|\text{grad } \phi|}{|\phi_x|} d\xi d\eta$$

$$= \iint_{B'} \frac{|\text{grad } \phi|}{|\phi_x|} |J| dxdy,$$

where J is the Jacobian of the mapping $\psi(x, t)$. Since J is equal to ϕ_x, we obtain

$$\int m(\phi^{-1}(t) \cap B')dt = \iint_{B'} |\text{grad } \phi| dxdy.$$

The same is true if $\phi_y \neq 0$ at z_0. We cover $K - K_0$ by countably many disks $\Delta_1, \Delta_2, \ldots$ like Δ. Since

$$\int m(\phi^{-1}(t) \cap \Delta_1 \cap (K-K_0))dt = \iint_{\Delta_1 \cap (K-K_0)} |\operatorname{grad} \phi| dxdy,$$

$$\int m(\phi^{-1}(t) \cap (\Delta_2-\Delta_1) \cap (K-K_0))dt = \iint_{(\Delta_2-\Delta_1) \cap (K-K_0)} |\operatorname{grad} \phi| dxdy,$$

$$\ldots,$$

we derive

$$\int m(\phi^{-1}(t) \cap (K-K_0))dt = \iint_{K-K_0} |\operatorname{grad} \phi| dxdy.$$

Finally let us prove that $m(\phi^{-1}(t) \cap K_0)$ is a measurable function of t and that $\int m(\phi^{-1}(t) \cap K_0)dt = 0$. Given $\varepsilon > 0$, choose $r > 0$ so that $|\phi(z') - \phi(z)| < \varepsilon|z' - z|$ whenever $z, z' \in K_0$ and $|z' - z| < r$. Divide K_0 into mutually disjoint Borel sets $B_1, \ldots, B_n$ of diameter less than r. By Lemma 22

$$\overline{\int} m(\phi^{-1}(t) \cap B_k)dt \leqq \frac{4\varepsilon}{m} m_2(B_k), \qquad k = 1, \ldots, n.$$

Therefore

$$\overline{\int} m(\phi^{-1}(t) \cap K_0)dt \leqq \sum_{k=1}^{n} \overline{\int} m(\phi^{-1}(t) \cap B_k)dt \leqq \frac{4\varepsilon}{m} m_2(K_0).$$

Since ε may be arbitrarily small, we conclude $\overline{\int} m(\phi^{-1}(t) \cap K_0)dt = 0$. It follows that $m(\phi^{-1}(t) \cap K_0)$ is a measurable function of t; in fact, it vanishes a.e. By taking a sum we derive (34). Thus (19) is proved.

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